The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2018–19**

Topics in Advanced Fluid Mechanics **2 hours 30 minutes**

Marks will be awarded for your best ***four*** *answers.*

1 (i) Starting from the 3D Euler equations for an incompressible fluid of a constant unit density

$$\frac{\partial \boldsymbol{u}}{\partial t} + (\boldsymbol{u} \cdot \nabla)\boldsymbol{u} = -\nabla p,$$

$$\nabla \cdot \boldsymbol{u} = 0,$$

derive the following set of equations for the impulse defined by $\boldsymbol{\gamma} = \boldsymbol{u} + \nabla\phi$,

$$\frac{D\boldsymbol{\gamma}}{Dt} = -(\nabla\boldsymbol{u})^T\boldsymbol{\gamma} + \nabla\lambda,$$

$$\frac{D\phi}{Dt} = p - \frac{|\boldsymbol{u}|^2}{2} + \lambda,$$

where T denotes matrix transpose and λ is an arbitrary scalar function of $\boldsymbol{x}$ and t. ***(19 marks)***

(ii) By using the impulse equations for a special choice of $\lambda = 0$ (geometric gauge) and the vorticity equations

$$\frac{D\boldsymbol{\omega}}{Dt} = \boldsymbol{\omega} \cdot \nabla\boldsymbol{u},$$

show that $\boldsymbol{\gamma} \cdot \boldsymbol{\omega}$ is conserved

$$\frac{D}{Dt}(\boldsymbol{\gamma} \cdot \boldsymbol{\omega}) = 0.$$

(6 marks)

2 We consider the Burgers equation in $\mathbb{R}^1$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} = \nu\frac{\partial^2 u}{\partial x^2}, \tag{1}$$

with an initial condition $u(x,0) = u_0(x)$. We also consider the heat diffusion equation in $\mathbb{R}^1$

$$\frac{\partial \psi}{\partial t} = \nu\frac{\partial^2 \psi}{\partial x^2}, \tag{2}$$

with an initial condition $\psi(x,0) = \psi_0(x)$.

(i) By direct computation, show that the following function

$$G(x,t) = \frac{1}{\sqrt{4\pi\nu t}}\exp\left(-\frac{x^2}{4\nu t}\right)$$

is a solution of (2) for $t > 0$. ***(8 marks)***

(ii) Show that the following function

$$\psi(x,t) = \psi_0 * G \equiv \int_{-\infty}^{\infty} \psi_0(y)G(x-y,t)dy$$

solves the initial-value problem (2), where * denotes a convolution product.

Hint: $G(x,t) \to \delta(x)$ as $t \to 0$, where $\delta(x)$ is Dirac delta function.
(4 marks)

(iii) State the Cole-Hopf transformation. ***(3 marks)***

(iv) By using the Cole-Hopf transformation, show that (1) can be reduced to (2). ***(10 marks)***

3 Consider a model equation for vorticity defined in $\mathbb{R}^1$:

$$\frac{\partial \omega}{\partial t} = \omega H[\omega],$$

with an initial condition

$$\omega(x,0) = \omega_0(x).$$

Here $H[\omega]$ denotes a Hilbert transform

$$H[\omega](x) = \frac{1}{\pi} ⨍_{-\infty}^{\infty} \frac{\omega(y)}{x-y} dy,$$

with $⨍$ a principal-value integral.

(i) Show that the equation for $H[\omega]$ is

$$\frac{\partial H[\omega]}{\partial t} = \frac{1}{2}\left(H[\omega]^2 - \omega^2\right).$$

Hint: for any f and g,
$H[fg] = H[f]g + fH[g] + H[H[f]H[g]]$ and $H[H[f]] = -f$. ***(7 marks)***

(ii) Drive the following equation for $f = \omega + iH[\omega]$:

$$\frac{\partial f(x,t)}{\partial t} = -\frac{i}{2} f(x,t)^2.$$

(4 marks)

(iii) By solving the above equation for $f(x,t)$, derive the expressions

$$\omega(x,t) = \frac{\omega_0(x)}{(1-\frac{t}{2}H[\omega_0])^2 + (\frac{t}{2}\omega_0(x))^2}$$

and

$$H[\omega](x,t) = \frac{H[\omega_0](1-\frac{t}{2}H[\omega_0]) - \frac{t}{2}\omega_0(x)^2}{(1-\frac{t}{2}H[\omega_0])^2 + (\frac{t}{2}\omega_0(x))^2}.$$

(7 marks)

(iv) Consider an initial condition $\omega_0(x) = -\dfrac{x}{x^2+a^2}$, where $a(\neq 0)$ is a constant. Using the exact solutions above, find an explicit formula for $\omega(x,t)$ for this initial condition. State when and where the breakdown (i.e. singularity formation) takes place and give an asymptotic form of $\omega(x,t)$ at the time of breakdown.

$$\text{Hint: } H\left[\frac{a}{x^2+a^2}\right] = \frac{x}{x^2+a^2},\ H\left[\frac{x}{x^2+a^2}\right] = -\frac{a}{x^2+a^2},$$

(7 marks)

4 We consider the Navier-Stokes equations subject to an external straining flow. The velocity field reads

$$\boldsymbol{u} = (-ax + u(x,y,t))\,\boldsymbol{i} + (-ay + v(x,y,t))\,\boldsymbol{j} + 2az\boldsymbol{k}, \quad a > 0,$$

where $\boldsymbol{i}, \boldsymbol{j}, \boldsymbol{k}$ denote the standard Cartesian unit vectors. We assume that the solutions u and v are functions of $r(= \sqrt{x^2 + y^2})$ and t only. In terms of the vorticity field

$$\boldsymbol{\omega} = \omega\boldsymbol{k}, \; \omega = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y},$$

the governing equation reads

$$\frac{\partial \omega}{\partial t} + (-ax + u)\frac{\partial \omega}{\partial x} + (-ay + v)\frac{\partial \omega}{\partial y} = 2a\omega + \nu \triangle \omega,$$

where

$$\triangle \omega \equiv \frac{\partial^2 \omega}{\partial r^2} + \frac{1}{r}\frac{\partial \omega}{\partial r}.$$

(i) Show that the equation for a steady solution can be written

$$-a\frac{1}{r}\frac{d}{dr}(r^2\omega) = \nu\frac{1}{r}\frac{d}{dr}\left(r\frac{d\omega}{dr}\right). \tag{1}$$

(8 marks)

(ii) By solving (1) under the boundary condition $\lim_{r\to\infty} \omega(r) = 0$, derive the following solution

$$\omega(r) = \frac{a\Gamma}{2\pi\nu}\exp\left(-\frac{ar^2}{2\nu}\right), \tag{2}$$

where $\Gamma = \int_{\mathbb{R}^2} \omega dx dy = \int_0^\infty \omega(r) 2\pi r dr$ denotes a constant. ***(10 marks)***

(iii) The energy dissipation rate, per unit length along the z-axis, is defined by

$$E = \nu \int_0^\infty \omega(r)^2 2\pi r dr.$$

Compute E for the above solution (2) and show that it is actually independent of ν. ***(7 marks)***

5 Consider a vortex layer of uniform strength, subject to a velocity field $-Ay\boldsymbol{j}$, where A denotes a constant and $\boldsymbol{j}$ the unit vector along the y-axis. Its motion is governed by

$$\frac{\partial z(\alpha,t)^*}{\partial t} = -\frac{i}{2\pi} ⨍_{-\infty}^{\infty} \frac{d\beta}{z(\alpha,t) - z(\beta,t)} + iA\,\mathrm{Im}(z(\alpha,t)), \tag{1}$$

where

$$z(\alpha,t) = x(\alpha,t) + iy(\alpha,t)$$

denotes the position of a fluid particle α on the layer, $*$ complex conjugate, Im the imaginary part and $⨍_{-\infty}^{\infty} = \lim_{L\to\infty} ⨍_{-L}^{L}$ a principal-value integral. We study the stability property of the vorticity layer around a flat state $z_0(\alpha) = \alpha$.

(i) By setting $z(\alpha) = \alpha + if(\alpha,t)$, derive from (1) that

$$\frac{\partial f(\alpha)^*}{\partial t} = \frac{1}{2\pi} ⨍_{-\infty}^{\infty} \frac{d\beta}{\alpha - \beta + i(f(\alpha) - f(\beta))} - Af(\alpha). \tag{2}$$

(3 marks)

(ii) Assuming $|f|$ to be small, derive the linearised equation from (2)

$$\frac{\partial f(\alpha)^*}{\partial t} = -\frac{i}{2\pi} ⨍_{-\infty}^{\infty} \frac{f(\alpha) - f(\beta)}{(\alpha - \beta)^2} d\beta - Af(\alpha). \tag{3}$$

(6 marks)

(iii) Assuming that $|f|$ is bounded, by integrating by parts show that (3) can be rewritten as

$$\frac{\partial f^*}{\partial t} = -\frac{i}{2} H\left[\frac{\partial f}{\partial \alpha}\right] - Af, \tag{4}$$

where H denotes the Hilbert transform (see **Question** 3 for definition).

(5 marks)

(iv) Derive from (4) that

$$\frac{\partial^2 f}{\partial t^2} = -\frac{1}{4}\frac{\partial^2 f}{\partial \alpha^2} + A^2 f.$$

(6 marks)

(v) Consider a Fourier mode of the form

$$f = \exp(ik\alpha + \lambda t),$$

where k is the wavenumber and λ the growth rate. Derive the dispersion relation, that is, an expression for λ as a function of k. State whether $A(\neq 0)$ stabilises or destabilises the vortex layer. ***(5 marks)***

End of Question Paper