Data provided: Formula Sheet

MAS420S

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — **Spring Semester 2018–2019**

MAS420S Signal Processing — **2 hours**

*Attempt **ALL** questions.*

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1 (i) In the Hilbert space of finite power signals of period T with inner product

$$(f,g) = \frac{1}{T}\int_{-\frac{T}{2}}^{\frac{T}{2}} f(t)g^*(t)dt,$$

where * denotes complex conjugate, the set $\phi_n(t) = e^{in\sigma t} : -\infty < n < \infty$, where $\sigma = \frac{2\pi}{T}$, forms an orthonormal basis. Hence any finite power signal, $f(t)$, can be written as $f(t) = \sum_{n=-\infty}^{\infty} c_n\phi_n(t)$. Use orthonormality to prove that the power, P, in $f(t)$ can be written

$$P = \sum_{n=-\infty}^{\infty} |c_n|^2.$$

(4 marks)

(ii) Using the frequency shift property of Fourier transforms, show that, if $f(t) \leftrightarrow F(\omega)$, then

$$f(t)\cos(\omega_0 t) \leftrightarrow \frac{1}{2}[F(\omega-\omega_0) + F(\omega+\omega_0)].$$

Using this result (or otherwise), show that

$$g(t) = p_{\frac{1}{2}}(t)\cos\pi t \leftrightarrow \frac{1}{2}\left[\operatorname{sinc}\left(\frac{1}{2}(\omega-\pi)\right) + \operatorname{sinc}\left(\frac{1}{2}(\omega+\pi)\right)\right]$$

(4 marks)

(iii) Determine the period and sketch a few periods (centred on $t = 0$) of the signal

$$g_1(t) = \sum_{n=-\infty}^{\infty} g(t-n),$$

where $g(t)$ is the signal defined in part (ii). Using the fact that a periodic signal of period T, $f_T(t)$, has complex Fourier coefficients $c_n = \frac{1}{T}F(n\sigma)$, where $F(\omega)$ is the Fourier transform of $f_T(t)p_{\frac{T}{2}}(t)$ and $\sigma = \frac{2\pi}{T}$, show that $g_1(t)$ has complex Fourier coefficients given by

$$c_n = \frac{2}{\pi}\frac{(-1)^{n+1}}{(4n^2-1)}.$$

(8 marks)

(iv) The signal $g_1(t)$ is passed through an ideal low-pass filter, $p_{3\pi}(\omega)$, to yield a signal $h(t)$. Write down the Fourier series for $h(t)$. Show that $> 99\%$ of the power in $g_1(t)$ is passed through this filter. ***(9 marks)***

2 (i) Sketch the 2-sided exponential pulse given by $f(t) = e^{-2|t|}$, $-\infty < t < \infty$. Use direct integration to show that the Fourier transform of $f(t)$ is $\dfrac{4}{4+\omega^2}$.

(5 marks)

(ii) With the aim of clear diagrams and without using Fourier Transforms show that the convolution of $f(t)$ with the rectangular pulse $p_a(t)$ for $t < -a$ is given by $e^{2t}\dfrac{(e^{2a} - e^{-2a})}{2}$ and find the convolution of $f(t)$ with $p_a(t)$ for all other values of t. Write down your full solution for the ranges $|t| < a$ and $|t| > a$.

(15 marks)

(iii) Hence deduce the Fourier transform of the signal $g(t) = \dfrac{8}{4+t^2}\,\mathrm{sinc}\, t$.

(5 marks)

3 (i) Define the following:

- a linear shift invariant (LSI) system;
- the system transfer function (STF), without any reference to the Fourier transform or the impulse response function;
- the impulse response function (IRF), without reference to the STF or convolution.

(6 marks)

(ii) A signal, $f(t)$, is input to a LSI system with system transfer function $H(\omega)$ giving an output signal

$$\int_{t-\frac{T}{2}}^{t+\frac{T}{2}} f(s)ds.$$

Show that $H(\omega) = T\,\mathrm{sinc}(\omega T/2)$. Find the response of the system to the signal $\delta(t)$ and show that it can be written as $p_{\frac{T}{2}}(t)$. ***(8 marks)***

(iii) Consider the system shown in Fig. 1 in which $f(t)$ is split into two branches containing components with STF $H(\omega) = T\,\mathrm{sinc}(\omega T/2)$ (the integrator in part (ii)) and another component with STF $J(\omega) = \alpha e^{-i\omega t_0}$. The outputs from each branch are summed to give the total output signal, $k(t)$.

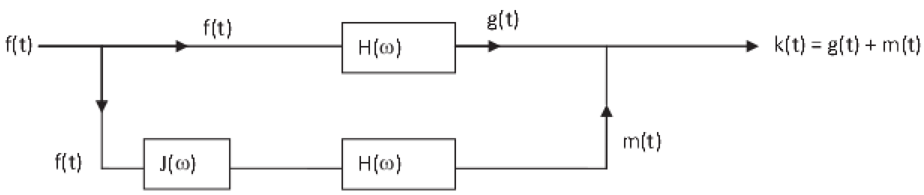

Fig. 1

(a) Explain why the STF for the whole system is given by

$$H(\omega)(1 + J(\omega)).$$

(b) Find the IRF of the system and hence find the input signal and the values of α and t_0 that would provide the output signal illustrated in Fig. 2.

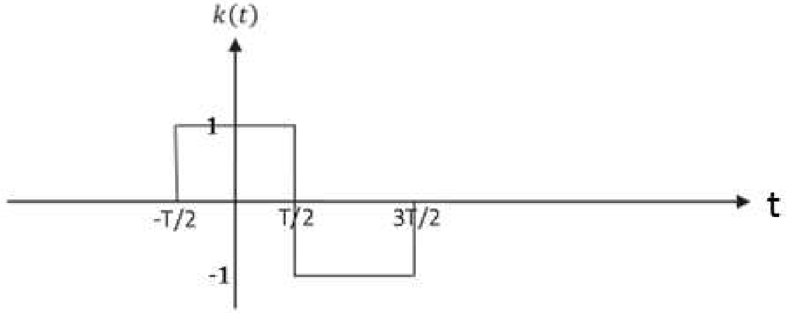

Fig. 2

(11 marks)

4 (i) For a signal $f(t)$ that is continuous at $t = 0$, show that

$$f(0) = \frac{1}{2\pi} \int_{-\infty}^{\infty} F(\omega)\, d\omega,$$

where $F(\omega)$ is the Fourier transform of $f(t)$. ***(2 marks)***

(ii) Make a sketch of $|f(t)|^2$ for a signal $f(t)$ with finite energy, E, and for which $|f(0)|^2 \geq |f(t)|^2$. Using your diagram, define the equivalent rectangle resolution,

$$\tau = \frac{E}{|f(0)|^2}.$$

(4 marks)

(iii) Consider the signal $f(t) = 3\,\mathrm{sinc}^2(2t)$. Calculate and sketch the signal spectrum, $F(\omega)$, and find

(a) its bandwidth, Ω (rad/s),

(b) its energy,

(c) the equivalent rectangle resolution, τ.

Verify that this signal satisfies the time-bandwidth theorem i.e. that $\tau\Omega > \pi$. ***(9 marks)***

(iv) If a signal $f(t)$ with Fourier transform, $F(\omega)$ is sampled with frequency $\frac{1}{T}$ Hz, the Fourier transform of the sampled signal is given by

$$F_S(\omega) = \frac{1}{T} \sum F(\omega - n\sigma),$$

where $\sigma = \frac{2\pi}{T}$ and $f(t) \leftrightarrow F(\omega)$. After sinc interpolation to obtain a signal $g(t)$, the Fourier transform, $G(\omega)$ is given by $G(\omega) = Tp_{\sigma/2}(\omega)F_S(\omega)$.

Find the Nyquist frequency (in Hz) for the signal in part (iii).

Sketch $F_S(\omega)$ for the case where the sampling frequency is half the Nyquist frequency. The samples are used to form a signal $g(t)$ using sinc interpolation. Use your sketch to determine the Fourier transform $G(\omega)$, of the recovered signal and hence find g(t). ***(10 marks)***

End of Question Paper

Function Definitions:

Rectangular pulse:

$$p_a(t) = \begin{cases} 1 & |t| \le a \\ 0 & |t| > a \end{cases}$$

Triangular pulse:

$$q_a(t) = \begin{cases} 1 - |t|/a & |t| \le a \\ 0 & |t| > a \end{cases}$$

Step function:

$$U(t) = \begin{cases} 1 & t \ge 0 \\ 0 & t < 0 \end{cases}$$

Fourier Transform Pairs:

$$\begin{aligned}
p_a(t) &\longleftrightarrow 2a\,\mathrm{sinc}(a\omega) \\
q_a(t) &\longleftrightarrow a\,\mathrm{sinc}^2(a\omega/2) \\
\mathrm{sinc}(at) &\longleftrightarrow \frac{\pi}{a}p_a(\omega) \\
\mathrm{sinc}^2(at) &\longleftrightarrow \frac{\pi}{a}q_{2a}(\omega) \\
\mathrm{e}^{-at}U(t) &\longleftrightarrow \frac{1}{a+i\omega} \\
\delta(t) &\longleftrightarrow 1 \\
\delta(t-t_0) &\longleftrightarrow \mathrm{e}^{-i\omega t_0} \\
1 &\longleftrightarrow 2\pi\delta(\omega) \\
\mathrm{e}^{i\omega_0 t} &\longleftrightarrow 2\pi\delta(\omega-\omega_0) \\
\mathrm{e}^{-t^2/2\sigma^2} &\longleftrightarrow \sigma\sqrt{2\pi}\mathrm{e}^{-\sigma^2\omega^2/2}
\end{aligned}$$

Fourier transform:

$$F(\omega) = \int_{-\infty}^{\infty} f(t)\mathrm{e}^{-i\omega t}\,\mathrm{d}t$$

Inverse Fourier transform:

$$f(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)\mathrm{e}^{i\omega t}\,\mathrm{d}\omega$$

Duality theorem: If $f(t) \longleftrightarrow F(\omega)$ then $F(t) \longleftrightarrow 2\pi f(-\omega)$

Scaling: If $f(t) \longleftrightarrow F(\omega)$ then $f(at) \longleftrightarrow \frac{1}{|a|}F(\omega/a)$.

Translation: If $f(t) \longleftrightarrow F(\omega)$ then $f(t-t_0) \longleftrightarrow \mathrm{e}^{-i\omega t_0}F(\omega)$.

Frequency Shift: If $f(t) \longleftrightarrow F(\omega)$ then $\mathrm{e}^{i\omega_0 t}f(t) \longleftrightarrow F(\omega-\omega_0)$

Fourier Series: If $f_T(t)$ is periodic with period T then, with $\sigma = 2\pi/T$, the complex Fourier series is

$$f_T(t) = \sum_{n=-\infty}^{\infty} c_n \mathrm{e}^{in\sigma t}$$

where

$$c_n = \frac{1}{T}\int_{-T/2}^{T/2} f_T(t)\mathrm{e}^{-in\sigma t}\ \mathrm{d}t$$

Likewise, the real Fourier series is

$$f_T(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}(a_n \cos n\sigma t + b_n \sin n\sigma t)$$

where

$$a_n = \frac{2}{T}\int_{-T/2}^{T/2} f_T(t)\cos n\sigma t\ \mathrm{d}t$$

$$b_n = \frac{2}{T}\int_{-T/2}^{T/2} f_T(t)\sin n\sigma t\ \mathrm{d}t$$

Parseval's Theorem: If V is a Hilbert space, $\{\phi_n\}$ is an orthonormal basis for V and $f = \sum_n c_n\phi_n$, then

$$\|f\|^2 = \sum_{n=-\infty}^{\infty} |c_n|^2$$

Plancherel's Theorem: If $f(t) \longleftrightarrow F(\omega)$ and $g(t) \longleftrightarrow G(\omega)$ then

$$\int_{-\infty}^{\infty} f(t)g^*(t)\ \mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)G^*(\omega)\ \mathrm{d}\omega$$

Energy Theorem: If $f(t) \longleftrightarrow F(\omega)$ then

$$\int_{-\infty}^{\infty} |f(t)|^2\ \mathrm{d}t = \frac{1}{2\pi}\int_{-\infty}^{\infty} |F(\omega)|^2\ \mathrm{d}\omega$$

Convolution Theorem: If $f(t) \longleftrightarrow F(\omega)$ and $g(t) \longleftrightarrow G(\omega)$ then

$$f(t) * g(t) = \int_{-\infty}^{\infty} f(s)g(t-s)\ \mathrm{d}s \longleftrightarrow F(\omega)G(\omega)$$

Product Theorem: If $f(t) \longleftrightarrow F(\omega)$ and $g(t) \longleftrightarrow G(\omega)$ then

$$f(t)g(t) \longleftrightarrow \frac{1}{2\pi}F(\omega) * G(\omega).$$

`[fsheet.tex: 30th August 2018, Rev. 1.2]`