SCHOOL OF MATHEMATICS AND STATISTICS

Analytic Number Theory

2 hours 30 minutes

Attempt all the questions. The allocation of marks is shown in brackets.

Note that the questions do not carry equal marks: Q1 is worth 29 marks, Q2 is worth 18 marks, Q3 is worth 20 marks, and Q4 is worth 33 marks.

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Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student
1  (i) State Bertrand’s Postulate. (2 marks)

Let $n \geq 4$. Using Bertrand’s Postulate, show that

(a) the sum $\sum_{k=1}^{n} \frac{1}{k}$ is not an integer, (5 marks)

(b) $n!$ is not a square. (3 marks)

(ii) Let $f, g : \mathbb{R} \to \mathbb{R}$ be two functions. Define what it means for $f$ and $g$ to be asymptotic. Define the prime counting function $\pi(x)$ and state the Prime Number Theorem. (3 marks)

(iii) Fix a positive integer $k$. For $x \geq 1$ let $\pi_k(x)$ be the number of primes $p$ such that $p^k \leq x$.

(a) Using the Prime Number Theorem, show that

$$\pi_k(x) \sim \frac{k \sqrt[3]{x}}{\ln x}.$$ (4 marks)

(b) For $0 < a < b$ evaluate

$$\lim_{x \to \infty} \frac{\pi_k(bx)}{\pi_k(ax)}.$$ 

Hence, or otherwise, prove that for $x$ large enough there exists a prime $p$ such that $ax < p^k \leq bx$. (8 marks)

(c) Using part (b) prove that there are infinitely many primes $p$ such that the decimal representation of $p^k$ begins with 5. (4 marks)
2. (i) (a) Let \( f, g_1, g_2 \) be arithmetic functions with \( f \) completely multiplicative. Show the following distribution property

\[
f \cdot (g_1 \ast g_2) = (f \cdot g_1) \ast (f \cdot g_2).
\]

Here (and in part (b) below) “\( \ast \)” refers to the usual pointwise multiplication of arithmetic functions. (3 marks)

(b) Hence show that, for \( a_1, a_2, a_3 \in \mathbb{Z} \),

\[
N_{a_1} \ast N_{a_2} \ast N_{a_3} = N_{a_3} \cdot (N_{a_1-a_3} \ast N_{a_2-a_3} \ast u),
\]

where \( N_\alpha(n) = n^\alpha \) and \( u(n) = 1 \) for all \( n \). (3 marks)

(ii) Write down the Euler product expansion of \( D(s, f) \) for a multiplicative arithmetic function \( f \). Hence show that if \( f \) is completely multiplicative then

\[
D(s, f) = \prod_p \left( 1 - \frac{f(p)}{p^s} \right)^{-1}.
\]

(4 marks)

(iii) (a) Show the identity

\[
1 + 4x + 9x^2 + 16x^3 + \ldots = \frac{1 - x^2}{(1 - x)^4}.
\]

(3 marks)

(b) Recall the arithmetic function \( \sigma_0(n) := \sum_{d|n} 1 \). Using the fact that \( \sigma_0 \) is multiplicative and part (a) above, prove the formal identity:

\[
D(s, \sigma_0^2) = \frac{\zeta(s)^4}{\zeta(2s)}.
\]

(5 marks)
(i) Recall that the Riemann zeta function \( \zeta(s) \) is defined for \( \text{Re}(z) > 1 \) by
\[
\zeta(s) := \sum_{n \geq 1} \frac{1}{n^s}.
\]

(a) Write down the Euler product for \( \zeta(s) \), indicating clearly in what region of the complex plane the formula is valid. (2 marks)

(b) The series
\[
\eta(s) := \sum_{n \geq 1} \frac{(-1)^{n+1}}{n^s}
\]
is convergent and analytic for \( \text{Re}(s) > 0 \). Derive a relation between \( \eta(s) \) and \( \zeta(s) \). Using this, write down, without proof, the analytic continuation of \( \zeta(s) \) to \( \text{Re}(s) > 0 \). (8 marks)

(c) State the Riemann Hypothesis. (2 marks)

(ii) Recall that \( B_n(x) \), the \( n \)-th Bernoulli polynomial, is defined by the generating series
\[
\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.
\]
Show that \( B_n(x + 1) - B_n(x) = nx^{n-1} \) for all \( n \geq 0 \). Deduce that
\[
1^{n-1} + 2^{n-1} + \ldots + N^{n-1} = \frac{B_n(N + 1) - B_n(1)}{n}
\]
for all integers \( n, N \geq 1 \). (8 marks)
(i) Let $\chi$ be a Dirichlet character mod $k$. Define the Dirichlet $L$-series $L(s, \chi)$. Describe a region where $L(s, \chi)$ is analytic and convergent, dependent on $\chi$. (3 marks)

(ii) For real $a > 0$ and $\text{Re}(s) > 1$, the Hurwitz zeta function is given by

$$\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}.$$

(a) Show that

$$\zeta \left( s, \frac{1}{2} \right) = (2^s - 1)\zeta(s).$$

(4 marks)

(b) If $\chi$ is a Dirichlet character mod $k$, then show that

$$L(s, \chi) = k^{-s} \sum_{r=0}^{k-1} \chi(r) \zeta \left( s, \frac{r}{k} \right).$$

(5 marks)

(iii) (a) List all characters of $(\mathbb{Z}/8\mathbb{Z})^\times$. (5 marks)

(b) For the nontrivial characters in your list, show explicitly the corresponding Dirichlet $L$-series does not vanish at $s = 1$. (3 marks)

(c) Verify using part (a) that for all primes $p$:

$$\sum_{\chi} \chi(5)^{-1} \chi(p) = \begin{cases} 4 & \text{if } p \equiv 5 \text{ mod } 8 \\ 0 & \text{otherwise} \end{cases}$$

where the sum is over all characters $\chi$ of $(\mathbb{Z}/8\mathbb{Z})^\times$. (4 marks)

(iv) Prove that there are infinitely many primes congruent to 5 mod 8.

(You may assume that for any character $\chi$ the sum $\sum_{p \nmid 8} \sum_{n \geq 2} \frac{\chi(p)^n}{np^{ns}}$ converges to a finite limit as $s \to 1$ and that $L(1, \chi) \neq 0$.) (9 marks)

End of Question Paper