The University Of Sheffield.

**SCHOOL OF MATHEMATICS AND STATISTICS** **Autumn Semester 2018–19**

**Analytic Number Theory** **2 hours 30 minutes**

*Attempt all the questions. The allocation of marks is shown in brackets.*

*Note that the questions do not carry equal marks: Q1 is worth 29 marks, Q2 is worth 18 marks, Q3 is worth 20 marks, and Q4 is worth 33 marks.*

**Please leave this exam paper on your desk**
**Do not remove it from the hall**

Registration number from U-Card (9 digits)
to be completed by student

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | | | | | |

Blank

**1** (i) State Bertrand's Postulate. ***(2 marks)***

Let $n \geq 4$. Using Bertrand's Postulate, show that

(a) the sum $\displaystyle\sum_{k=1}^{n} \frac{1}{k}$ is not an integer, ***(5 marks)***

(b) $n!$ is not a square. ***(3 marks)***

(ii) Let $f, g : \mathbb{R} \to \mathbb{R}$ be two functions. Define what it means for $f$ and $g$ to be asymptotic. Define the prime counting function $\pi(x)$ and state the Prime Number Theorem. ***(3 marks)***

(iii) Fix a positive integer $k$. For $x \geq 1$ let $\pi_k(x)$ be the number of primes $p$ such that $p^k \leq x$.

(a) Using the Prime Number Theorem, show that

$$\pi_k(x) \sim \frac{k\sqrt[k]{x}}{\ln x}.$$

***(4 marks)***

(b) For $0 < a < b$ evaluate

$$\lim_{x\to\infty} \frac{\pi_k(bx)}{\pi_k(ax)}.$$

Hence, or otherwise, prove that for $x$ large enough there exists a prime $p$ such that $ax < p^k \leq bx$. ***(8 marks)***

(c) Using part (b) prove that there are are infinitely many primes $p$ such that the decimal representation of $p^k$ begins with 5. ***(4 marks)***

**2** (i) (a) Let $f, g_1, g_2$ be arithmetic functions with $f$ completely multiplicative. Show the following distribution property

$$f \cdot (g_1 \star g_2) = (f \cdot g_1) \star (f \cdot g_2).$$

Here (and in part (b) below) " $\cdot$ " refers to the usual pointwise multiplication of arithmetic functions. ***(3 marks)***

(b) Hence show that, for $a_1, a_2, a_3 \in \mathbb{Z}$,

$$N_{a_1} \star N_{a_2} \star N_{a_3} = N_{a_3} \cdot (N_{a_1-a_3} \star N_{a_2-a_3} \star u),$$

where $N_\alpha(n) = n^\alpha$ and $u(n) = 1$ for all $n$. ***(3 marks)***

(ii) Write down the Euler product expansion of $D(s, f)$ for a multiplicative arithmetic function $f$. Hence show that if $f$ is completely multiplicative then

$$D(s, f) = \prod_p \left(1 - \frac{f(p)}{p^s}\right)^{-1}.$$

***(4 marks)***

(iii) (a) Show the identity

$$1 + 4x + 9x^2 + 16x^3 + ... = \frac{1 - x^2}{(1 - x)^4}.$$

***(3 marks)***

(b) Recall the arithmetic function $\sigma_0(n) := \sum_{d|n} 1$. Using the fact that $\sigma_0$ is multiplicative and part (a) above, prove the formal identity:

$$D(s, \sigma_0^2) = \frac{\zeta(s)^4}{\zeta(2s)}.$$

***(5 marks)***

**3** (i) Recall that the Riemann zeta function $\zeta(s)$ is defined for $\mathrm{Re}(z) > 1$ by

$$\zeta(s) := \sum_{n\geq 1} \frac{1}{n^s}.$$

(a) Write down the Euler product for $\zeta(s)$, indicating clearly in what region of the complex plane the formula is valid. ***(2 marks)***

(b) The series

$$\eta(s) := \sum_{n\geq 1} \frac{(-1)^{n+1}}{n^s}$$

is convergent and analytic for $\mathrm{Re}(s) > 0$. Derive a relation between $\eta(s)$ and $\zeta(s)$. Using this, write down, without proof, the analytic continuation of $\zeta(s)$ to $\mathrm{Re}(s) > 0$. ***(8 marks)***

(c) State the Riemann Hypothesis. ***(2 marks)***

(ii) Recall that $B_n(x)$, the $n$-th Bernoulli polynomial, is defined by the generating series

$$\frac{te^{xt}}{e^t - 1} = \sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!}.$$

Show that $B_n(x+1) - B_n(x) = nx^{n-1}$ for all $n \geq 0$. Deduce that

$$1^{n-1} + 2^{n-1} + \ldots + N^{n-1} = \frac{B_n(N+1) - B_n(1)}{n}$$

for all integers $n, N \geq 1$. ***(8 marks)***

**4** (i) Let $\chi$ be a Dirichlet character mod $k$. Define the Dirichlet $L$-series $L(s, \chi)$. Describe a region where $L(s, \chi)$ is analytic and convergent, dependent on $\chi$. ***(3 marks)***

(ii) For real $a > 0$ and $\text{Re}(s) > 1$, the Hurwitz zeta function is given by

$$\zeta(s, a) = \sum_{n=0}^{\infty} \frac{1}{(n+a)^s}.$$

(a) Show that

$$\zeta\left(s, \frac{1}{2}\right) = (2^s - 1)\zeta(s).$$

***(4 marks)***

(b) If $\chi$ is a Dirichlet character mod $k$, then show that

$$L(s, \chi) = k^{-s} \sum_{r=0}^{k-1} \chi(r)\zeta\left(s, \frac{r}{k}\right).$$

***(5 marks)***

(iii) (a) List all characters of $(\mathbb{Z}/8\mathbb{Z})^{\times}$. ***(5 marks)***

(b) For the nontrivial characters in your list, show explicitly the corresponding Dirichlet $L$-series does not vanish at $s = 1$. ***(3 marks)***

(c) Verify using part (a) that for all primes $p$:

$$\sum_{\chi} \chi(5)^{-1}\chi(p) = \begin{cases} 4 & \text{if } p \equiv 5 \bmod 8 \\ 0 & \text{otherwise} \end{cases}$$

where the sum is over all characters $\chi$ of $(\mathbb{Z}/8\mathbb{Z})^{\times}$. ***(4 marks)***

(iv) Prove that there are infinitely many primes congruent to 5 mod 8.

(You may assume that for any character $\chi$ the sum $\displaystyle\sum_{p \nmid 8} \sum_{n \geq 2} \frac{\chi(p)^n}{np^{ns}}$ converges to a finite limit as $s \to 1$ and that $L(1, \chi) \neq 0$.) ***(9 marks)***

## End of Question Paper