SCHOOL OF MATHEMATICS AND STATISTICS

Autumn Semester
2018-19

Fields

2 hours 30 minutes

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) Let L be a field and K be a subset of L. What is meant by saying that K is a subfield of L? (3 marks)

(ii) For each of the subsets J_1 and J_2 of \( \mathbb{C} \) specified below determine, with justification, whether it is a subfield of \( \mathbb{C} \):

(a) \( J_1 = \{a + b\sqrt{7} : a, b \in \mathbb{Q}\} \). (5 marks)

(b) \( J_2 = \{a + b\sqrt{3} + i\sqrt{3} : a, b, c \in \mathbb{Q}\} \) where \( i = \sqrt{-1} \). (5 marks)

(iii) (iii) Let \( L = \mathbb{Q}(\sqrt{3}, \sqrt{5}) \) and \( a = p\sqrt{3} + q\sqrt{5} \) where p and q are positive rational numbers.

(a) Show that \( L = \mathbb{Q}(a) \). (5 marks)

(b) Let \( b = p\sqrt{3} - q\sqrt{5} \). Show that \( b \neq 0 \). (3 marks)

(c) Express the element \( b^{-1} \) as a sum \( \lambda_0 + \lambda_1 a + \lambda_2 a^2 + \lambda_3 a^3 \) where \( \lambda_i \in \mathbb{Q} \). (4 marks)
2  (i)  Give a definition of n-th \textit{cyclotomic polynomial} $\phi_n(x)$ where n is a natural number. \hspace{1cm} (2 marks)

(ii) Prove that

$$\phi_p(x) = x^{p-1} + x^{p-2} + \cdots + x + 1$$

provided $p$ is a prime number. \hspace{1cm} (4 marks)

(iii) Give an explicit expression for the polynomial $\phi_n(x)$ for $n = 1, 2, 3, 4$. \hspace{1cm} (4 marks)

(iv) Let $p$ be a prime number. Prove that

$$\phi_p(x) = x^{p-1} + x^{p-2} + \cdots + x + 1$$

is an irreducible polynomial in $\mathbb{Q}[x]$. \hspace{1cm} (8 marks)

(v) Let $K \subseteq L$ be a field extension, and let an element $\alpha \in L$ be algebraic over the field $K$.

(a) Give the definition of the \textit{minimal polynomial} of $\alpha$ over $K$. \hspace{1cm} (2 marks)

(b) Show that the minimal polynomial of $\alpha$ over $K$ is an irreducible polynomial in $K[x]$. \hspace{1cm} (2 marks)

(c) Let $f$ be a monic irreducible polynomial with $f(\alpha) = 0$. Show that $f$ is the minimal polynomial of $\alpha$. \hspace{1cm} (3 marks)

3  (i) Let $K$ be a subfield of a field $L$. Give a definition of $[L : K]$. \hspace{1cm} (2 marks)

(ii) State the degrees formula for finite field extensions $K \subseteq L \subseteq M$. \hspace{1cm} (2 marks)

(iii) Prove the degrees formula for finite field extensions $K \subseteq L \subseteq M$. \hspace{1cm} (9 marks)

(iv) Let $K_0 \subseteq K_1 \subseteq \cdots \subseteq K_n$ be finite field extensions. Prove that

$$[K_n : K_0] = [K_n : K_{n-1}][K_{n-1} : K_{n-2}] \cdots [K_1 : K_0].$$

(5 marks)

(v) Using (iv) (or otherwise) find $[\mathbb{Q}(\sqrt{3}, \sqrt{5}, i) : \mathbb{Q}]$. \hspace{1cm} (7 marks)
4  (i) Let $K \subseteq L$ be fields. Define the group $G(L/K)$. \hfill (2 marks)

(ii) Let $K_1 = \mathbb{Q}(i)$ and $L = K_1(\sqrt[3]{5})$.
(a) Find $[L : K_1]$. \hfill (10 marks)
(b) Prove that the polynomial $m_1(x) = x^3 - 5$ is irreducible over the field $K_1$. \hfill (5 marks)
(c) Write the polynomial $m_1(x) = x^3 - 5$ as a product of irreducible monic polynomials over the field $L$. \hfill (6 marks)
(d) Find the group $G(L/K_1)$. \hfill (2 marks)

End of Question Paper