The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2018–19**

Financial Mathematics **2 hours and 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

Please leave this exam paper on your desk
Do not remove it from the hall

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to be completed by student

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1 (i) Consider the following three bonds with face value of £100:

Time to maturity (in years)	Annual interest (paid every 6 months)	Bond price (in £)
1.	10%	105.808
1.	6%	101.917
1.5	8%	104.995

(a) Write down a system of equations satisfied by the 0.5-year and 1-year spot interest rates. ***(4 marks)***

(b) Find the 0.5-year, 1-year and 1.5-year spot interest rates. ***(11 marks)***

(ii) Consider a twelve-month forward contract on one share of *XYZ Plc*. These shares are currently traded for £4 per share, and spot interest rates for all maturities are 5%. Within the next twelve months, *XYZ Plc* will pay a single dividend of 10p per share in 3 months.

(a) Find the present value of the dividend paid in 3 months and the twelve-month forward price of one share of *XYZ Plc*. ***(2 marks)***

(b) You are given the opportunity to take a long position in this forward contract at a forward price of £4. Describe in detail an arbitrage opportunity available to you. ***(8 marks)***

2 (i) Consider two European call options with the same underlying asset and same expiration date in T-years, and with strike prices $X_1 < X_2$. Consider also a portfolio which is long one call option with strike price X_1 and short one call option with strike X_2. Let r be the T-year spot interest rate.

(a) Sketch the payoff function of the portfolio. ***(6 marks)***

(b) Let c_1 and c_2 be the prices of the call options above with strike prices X_1 and X_2, respectively. Use part (a) to deduce that $c_1 - c_2 \leq e^{-rT}(X_2 - X_1)$. Justify your reasoning. ***(6 marks)***

(ii) The price of a stock which pays no dividends is currently £64 and over the next three 1-year periods the price will either increase by 25% or decrease by 50%. Interest rates for all periods are 5%.

(a) Use a binomial tree to find the price of an American put option on this stock with strike price £70. ***(11 marks)***

(b) When would a rational owner of this put option exercise it before expiration? ***(2 marks)***

3 (i) (a) State Ito's Lemma. ***(3 marks)***

(b) Let X_t denote the USD/GBP exchage rate at time $t \geq 0$ (i.e., X_t is the price in USD of £1 at time t). Assume that $\{X_t\}_{t\geq 0}$ is an Ito process given by $dX = \alpha X dt + \beta X dB$ for some constants α and β. Consider the stochastic process $Y = 1/X$ followed by the GBP/USD exchange rate. What is dY? ***(6 marks)***

(ii) Assume that a stock price S is given as the Ito process

$$dS = \mu S\, dt + \sigma S\, dB$$

where μ and σ are constants. Assume also that all interest rates are non-stochastic and equal to r.

In this question we assume the following: if $f(S,t)$ is a solution of the Black-Scholes PDE

$$\frac{\partial f}{\partial t} + rS\frac{\partial f}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 f}{\partial S^2} = rf,$$

then so is

$$\left(\frac{S}{K}\right)^{1-\frac{2r}{\sigma^2}} f(K^2/S, t)$$

for any constant K. (You do *not* need to prove this.)

(a) Let $c(S,t)$ denote the price at time $0 \leq t \leq T$ of an European call option on the underlying asset whose price is given by S, whose strike price is X and which expires at time T. Fix a constant $0 < K < X$ and define

$$v(S,t) = c(S,t) - \left(\frac{S}{K}\right)^{1-\frac{2r}{\sigma^2}} c(K^2/S, t).$$

Explain why is v also a solution of the Black-Scholes PDE. ***(3 marks)***

(b) Calculate $v(K,t)$ and show that $v(S,T) = c(S,T)$ if $S > K$. ***(5 marks)***

(c) A *knock-out barrier* option on the asset whose price is given by S, with strike price X and expiration time T pays owner at time T as follows. If $S_t \leq K$ at any time $0 \leq t \leq T$, the owner of this option gets nothing, otherwise the owner is paid $c(S_T, T)$.

Consider the price $w(S,t)$ of this option at time $0 \leq t < T$. Show that $w(S_t,t) = 0$ if $S_\tau \leq K$ for some $0 \leq \tau \leq t$, and that $w(S_t,t) = v(S_t,t)$ if $S_\tau > K$ for all $0 \leq \tau \leq t$. ***(8 marks)***

4 (i) Alice and Bob invest for the same period of time in a market which includes a risk-free investment and many other risky investments. Assume that the CAPM holds. For each one of the following statements, determine whether it is true or false and justify your claim. (For some of the items below you may want to use a sketch in the σ-r plane to support your answer.)

(a) Alice and Bob must hold the same proportions of all assets. ***(3 marks)***

(b) Alice and Bob will invest in the same risky assets. ***(3 marks)***

(c) When choosing between two portfolios, Alice and Bob always prefer the one with the lowest standard deviation of returns. ***(3 marks)***

(d) Alice must hold any two risky assets in the same ratio as Bob does in his portfolio. ***(3 marks)***

(ii) Consider a market with risk-free return r_B and whose market portfolio M has expected return r_M and standard deviation of returns σ_M. Let A be an investment with expected return of r_A, standard deviation of returns σ_A and beta coefficient β.

(a) What is the slope of the capital market line? ***(1 mark)***

(b) Describe parametrically the curve c in the σ-r plane consisting of all points corresponding to investments spread between A and M. ***(5 marks)***

(c) Explain why c is tangent to the capital market line at the point M. ***(3 marks)***

(d) Use (c) to show that

$$r_A = \beta(r_M - r_B) + r_B.$$ ***(4 marks)***

End of Question Paper