Attempt all the questions. The allocation of marks is shown in brackets.

Throughout the paper $K$ denotes a subfield of $\mathbb{C}$ which contains $\mathbb{Q}$.

All field extensions are finite.
(i) State, without proof, the Theorem of the Primitive Element (TPE).

(ii) Let $K \subset L$ be an extension of fields.
   (a) What does it mean for $\theta : L \to L$ to be a $K$-automorphism of $L$?

   (b) Let $\alpha \in L$ be such that $f(\alpha) = 0$ for some polynomial $f(x) \in K[x]$.
       If $\theta$ is a $K$-automorphism of $L$, show that $\theta(\alpha)$ is a root of $f(x)$
       too.

   (c) Define the Galois group $\text{Gal}(L/K)$ of the field extension $K \subset L$, and
       say what it means in terms of this group, for $K \subset L$ to be a Galois
       extension. Given an equivalent formulation involving splitting fields.

(iii) Let $K \subseteq M \subseteq L$ be finite extensions of fields. Suppose that $L/K$ is Galois.
   (a) Prove that $L/M$ is Galois.

   (b) If $M/K$ is Galois, prove that $\varphi(M) \subseteq M$ for all $\varphi \in \text{Gal}(L/K)$.

   (c) If $M/K$ is Galois, deduce that $\text{Gal}(L/M) \triangleleft \text{Gal}(L/K)$, and that
       $\text{Gal}(M/K) \cong \frac{\text{Gal}(L/K)}{\text{Gal}(L/M)}$.
2. (i) Let $K$ be a field, and let $f \in K[x]$ be a polynomial of degree $n$.
(a) Define the Galois group $\text{Gal}(f/K)$ of $f$. \hfill (1 mark)
(b) Show that there is an injective homomorphism
\[ \text{Gal}(f/K) \rightarrow S_n, \]
where $S_n$ denotes the symmetric group on $n$ letters. \hfill (7 marks)
(c) Deduce that the splitting field of a polynomial of degree $n$ over $K$ has degree at most $n!$ over $K$. \hfill (2 marks)

(ii) What are the Galois groups (up to isomorphism) for the following irreducible quartics over $\mathbb{Q}$?
(a) $x^4 - 3$. \hfill (4 marks)
(b) $x^4 + 1$. \hfill (3 marks)
(c) $x^4 + x^3 + x^2 + x + 1$. \hfill (2 marks)

3. Let $f = x^4 - 2x^2 - 6 \in \mathbb{Q}[x]$ and let $M$ denote the splitting field of $f$ over $\mathbb{Q}$. Let $\alpha = \sqrt{1 + \sqrt{7}}$.
(i) Show that the roots of $f$ are $\pm \alpha, \pm \frac{i\sqrt{6}}{\alpha}$, and deduce that $M = \mathbb{Q}(\alpha, i\sqrt{6})$. \hfill (4 marks)
(ii) It is given that $[M : \mathbb{Q}] = 8$. Specify the elements of $\text{Gal}(M/\mathbb{Q})$ by giving their effect on each of $\alpha$ and $i\sqrt{6}$, justifying your answer. \hfill (8 marks)
(iii) Show that there exist automorphisms $\varphi, \psi \in \text{Gal}(M/\mathbb{Q})$ such that $\varphi$ has order 4, $\psi$ has order 2, and $\text{Gal}(M/\mathbb{Q}) = \langle \varphi, \psi \rangle$. \hfill (5 marks)
(iv) Write $\psi \varphi \psi^{-1}$ in the form $\varphi^i \psi^j$. To which well-known group is $\text{Gal}(M/\mathbb{Q})$ isomorphic? \hfill (3 marks)
(v) Write $L = \mathbb{Q} \left( \alpha + \frac{i\sqrt{6}}{\alpha} \right)$. Using the Galois correspondence, find $[L : \mathbb{Q}]$. \hfill (5 marks)
(i) Let $a, b$ be two coprime positive integers that are not squares. Let $L = \mathbb{Q}(\sqrt{a}, \sqrt{b})$. Compute the Galois group $\text{Gal}(L/\mathbb{Q})$ and write down the effect of every element on $\sqrt{a}$ and $\sqrt{b}$. (3 marks)

(ii) Prove that the Galois group over $\mathbb{Q}$ of one of the polynomials $x^4 + x + \frac{3}{4}$ and $x^4 + x - \frac{3}{4}$ is $A_4$ and that the other is $S_4$. (7 marks)

[You may assume that the resolvent cubic for quartics of the form $x^4 + ax + b$ is given by $y^3 - 4by - a^2$, and that both have discriminant $256b^3 - 27a^4$.]

(iii) Show that $x^5 - 30x + 12$ over $\mathbb{Q}$ is not soluble by radicals by proving that its Galois group isomorphic to $S_5$. (Hint. You may use the following fact without proof: Any transitive subgroup of $S_5$ which contains a transposition is equal to $S_5$.) (13 marks)

End of Question Paper