The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2018–2019**

MAS6450 Waves and Magnetohydrodynamics **2 hours**

Answer all four questions. Formulae are on the last page.

Please leave this exam paper on your desk
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to be completed by student

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1 (i) A magnetic field is given by

$$\boldsymbol{B}(x, y, z) = B_0 \, (pxz, qyz, s)$$

where B_0, p, q and s are constants. What is the relationship between p and q?
(2 marks)

(ii) Consider a simple plasma flow $\boldsymbol{v} = v_0 \sin(ky)\hat{\mathbf{x}}$ with v_0 and k as constants. For a magnetic field lying in the $x - y$ plane,

$$\boldsymbol{B} = B_x\hat{\mathbf{x}} + B_y\hat{\mathbf{y}},$$

where $\hat{\mathbf{x}}$ and $\hat{\mathbf{y}}$ are unit vectors along x and y axis respectively,

(a) Write down the components of the ideal induction equation.
(3 marks)

(b) If the initial configuration is a uniform vertical field $\boldsymbol{B}(\boldsymbol{x}, 0) = B_0\hat{\mathbf{y}}$, what is the solution to the induction equation in (a). ***(3 marks)***

(c) Find the equation of magnetic field lines. (Hint: Use the solution obtained in (b)). ***(4 marks)***

 Question 1 continued on next page

1 (continued)

(iii) You are given a magnetic field, $\boldsymbol{B} = (0, B, 0)$ such that

$$\frac{\partial B}{\partial t} = \eta \frac{\partial^2 B}{\partial x^2}, \qquad (*)$$

where $\eta = \dfrac{1}{\mu_0 \sigma}$ is a positive constant.

(a) Using the equation $(*)$, show that the rate of change of magnetic energy

$$\frac{\partial}{\partial t} \int_{-\infty}^{\infty} \frac{B^2}{2\mu_0} dx$$

is negative. ***(6 marks)***

(b) Find the current density $\boldsymbol{J}$ and express the rate of magnetic energy decrease in terms of current density. ***(4 marks)***

(c) Comment on the change in magnetic energy and its relationship to ohmic heating. ***(1 mark)***

(iv) If $B_\theta(r) = B_0 r \mathrm{e}^{-r}$, sketch B_θ as a function of r, marking clearly the maximum point on the sketch. ***(2 marks)***

2 (i) Given a velocity field $\boldsymbol{v} = (yz, -xz, 0)$ and the initial magnetic field $\boldsymbol{B}(\boldsymbol{x}, 0) = (x, -y, 0)$, find $\boldsymbol{B}(\boldsymbol{x}, t)$ by obtaining the Lagrangian coordinates corresponding to $\boldsymbol{v}$ and applying the Cauchy solution. ***(20 marks)***

(ii) Verify that the field $\boldsymbol{B}(\boldsymbol{x}, t)$ obtained in (i) is indeed a solution, by direct substitution in the following ideal induction equation

$$\frac{\partial \boldsymbol{B}}{\partial t} = \nabla \times (\boldsymbol{v} \times \boldsymbol{B}),$$

(5 marks)

3 (i) Derive the Induction equation from the Maxwell's equation:

$$\frac{\partial \boldsymbol{B}}{\partial t} = -\nabla \times \boldsymbol{E},$$

using the Ohm's law:

$$\eta \boldsymbol{J} \quad = \quad \boldsymbol{E} + (\boldsymbol{v} \times \boldsymbol{B}),$$

and the Ampere's law:

$$\mu_0 \boldsymbol{J} \quad = \quad \nabla \times \boldsymbol{B}$$

(4 marks)

(ii) Show from Maxwell's equation that $\nabla \cdot \boldsymbol{B} = 0$ at all times if $\nabla \cdot \boldsymbol{B} = 0$ at $t = 0$. ***(3 marks)***

(iii) Sketch the magnetic field lines for the magnetic field $\boldsymbol{B} = B_0(x, -y)$, where B_0 is a positive constant. ***(5 marks)***

(iv) The linearised momentum equation for a fluid in a rotating frame of reference rotating with a uniform angular velocity $\boldsymbol{\Omega}$ is given by

$$\rho_0 \frac{\partial \boldsymbol{v}_1}{\partial t} + 2\rho_0 (\boldsymbol{\Omega} \times \boldsymbol{v}_1) = -\nabla \left(p_1 + \frac{\boldsymbol{B}_1 \cdot \boldsymbol{B}_0}{\mu_0} \right) + \frac{1}{\mu_0} (\boldsymbol{B}_0 \cdot \nabla) \boldsymbol{B}_1$$

where ρ_0 and B_0 are constants. Here, the subscripts "0" and "1" denote the equilibrium and perturbed quantities respectively and the parameters have the usual meaning (Note: the induction equation in a rotating frame of reference is the same as in an inertial frame). Seeking the plane wave solution of the form:

$$\boldsymbol{v}_1 \quad = \quad \hat{\mathbf{v}}_{\mathbf{1}} e^{i(\mathbf{k} \cdot \mathbf{x} - \omega t)}$$

etc. for the perturbed quantities, show that the dispersion relation for an ideal, incompressible, inviscid fluid can be given by the following equation

$$\omega^2 \quad = \quad \frac{\boldsymbol{B}_0 \cdot \boldsymbol{k}}{\mu_0 \rho_0} \mp \frac{2(\boldsymbol{k} \cdot \boldsymbol{\Omega})\omega}{k}$$

(13 marks)

4 (i) Show that the steady state solution of the induction equation for a flow $\boldsymbol{v} = V_0(x, -y, 0)$ interacting with a magnetic field $\boldsymbol{B} = B(x)\hat{\mathbf{y}}$ is given by

$$B(x) = B_0 \mathrm{e}^{-\frac{V_0 x^2}{2\eta}}$$

where V_0, B_0 and η are positive constants. Here, $\hat{\mathbf{y}}$ is the unit vector along the y axis. You may assume that $B^{'}(x) \to 0$ and $B(x) \to 0$ faster than x^{-1} as $|x| \to \infty$, where $'$ is differentiation with respect to x.

(Hint: Ignore the effect of magnetic field on the flow and thus, neglect the momentum equation. Only consider the induction equation). ***(8 marks)***

(ii) What is a force-free magnetic field? ***(1 mark)***

Show clearly that the magnetic field

$$\boldsymbol{B} = B_0(\sin kz, \cos kz, 0)$$

is a force-free field. ***(6 marks)***

(iii) (a) Consider a uniform vertical magnetic field in the presence of gravity. Thus,

$$\boldsymbol{B} = B_0\hat{\mathbf{z}}; \quad \boldsymbol{g} = -g\hat{\mathbf{z}}.$$

Using the hydrostatic pressure balance equation

$$\frac{dp}{dz} = -\rho(z)g,$$

show that the pressure, $p(z)$ and density $\rho(z)$ decrease exponentially for an isothermal plasma. ***(6 marks)***

(Hint: You can use the ideal gas law: $p(z) = \rho(z)RT$ where R is a gas constant and T is the temperature.)

(b) Define plasma beta, β. ***(2 marks)***

Which force terms in the momentum equation can be neglected if $\beta \ll 1$, and $\beta \gg 1$. ***(2 marks)***

5 Formulae Sheet

$$\mathbf{A}\times(\mathbf{B}\times\mathbf{C}) = (\mathbf{C}\times\mathbf{B})\times\mathbf{A} = (\mathbf{A}\cdot\mathbf{C})\mathbf{B} - (\mathbf{A}\cdot\mathbf{B})\mathbf{C}$$

$$\nabla^2\mathbf{A} = \nabla(\nabla\cdot\mathbf{A}) - \nabla\times\nabla\times\mathbf{A}$$

	u	v	w	f	g	h
cartesian	x	y	z	1	1	1
spherical	r	θ	ϕ	1	r	$r\sin\theta$
cylindrical	r	ϕ	z	1	r	1

$$\nabla\cdot\mathbf{V} = \frac{1}{fgh}\left[\frac{\partial}{\partial u}(ghV_u) + \frac{\partial}{\partial v}(fhV_v) + \frac{\partial}{\partial w}(fgV_w)\right]$$

$$\nabla\times\mathbf{V} = \frac{1}{gh}\left[\frac{\partial}{\partial v}(hV_w) - \frac{\partial}{\partial w}(gV_v)\right]\hat{u} + \frac{1}{fh}\left[\frac{\partial}{\partial w}(fV_u) - \frac{\partial}{\partial u}(hV_w)\right]\hat{v} + \frac{1}{fg}\left[\frac{\partial}{\partial u}(gV_v) - \frac{\partial}{\partial v}(fV_u)\right]\hat{w}$$

vector identity:

$$\nabla\times(\mathbf{F}\times\mathbf{G}) = \mathbf{F}(\nabla\cdot\mathbf{G}) - \mathbf{G}(\nabla\cdot\mathbf{F}) + (\mathbf{G}\cdot\nabla)\mathbf{F} - (\mathbf{F}\cdot\nabla)\mathbf{G}$$

End of Question Paper