The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2018–2019**

Mathematics Core II **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

This exam paper has two sections. Section A consists of multiple choice questions which must be answered on the exam paper itself.

Answers to Section B must be written on the answer booklet provided.

Total marks: ***55***

Please leave this exam paper on your desk
Do not remove it from the hall

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Section A

Each question or incomplete statement in this section is followed by four possible options of which exactly one is correct. Mark clearly the correct answer on the question paper. ***(22 marks)***

A1 Give the line perpendicular to $x = 2y + 1$ passing through $(1, 1)$.

A. $y = -\frac{1}{2}x + 1$ **B.** $2x + y = 3$ **C.** $2x = y + 1$ **D.** $y = \frac{1+x}{2}$

A2 Let P denote the point $(-1, \sqrt{3})$. In polar coordinates, what is the corresponding value of θ?

A. $\frac{\pi}{6}$ **B.** $\frac{2\pi}{3}$ **C.** $\frac{3\pi}{4}$ **D.** $\frac{7\pi}{6}$

A3 Which of the following matrices is in row echelon form?

A. $\begin{pmatrix} 1 & 1 & 3 & 4 \\ 0 & 2 & 1 & 3 \end{pmatrix}$ **B.** $\begin{pmatrix} 0 & 2 & 2 & 4 \\ 1 & 3 & 2 & 2 \end{pmatrix}$ **C.** $\begin{pmatrix} 1 & 2 & 1 & 3 \\ 2 & 5 & 0 & 4 \end{pmatrix}$ **D.** $\begin{pmatrix} 0 & 1 & 0 & 1 \\ 0 & 2 & 1 & 1 \end{pmatrix}$

A4 Three linear equations in the five variables x, y, z, t, u are solved using complete elimination, ending with the matrix $\left(\begin{array}{ccccc|c} 1 & 2 & 0 & 3 & 4 & 2 \\ 0 & 0 & 1 & 2 & 3 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right)$. Which variables are the easiest choices for parameters for the solutions?

A. x and y **B.** x and z **C.** x, y and z **D.** y, t and u

A5 A is a 2×3 matrix, B is a 3×1 matrix and C is a 2×3 matrix. Which of the following products make sense?

A. BA^T **B.** CA^T **C.** CB^T **D.** BC^T

A6 If $A = \begin{pmatrix} 1 & 2 \\ 3 & -1 \end{pmatrix}$ and $B = \begin{pmatrix} -1 & 1 \\ 2 & -3 \end{pmatrix}$, what is AB?

A. $\begin{pmatrix} 1 & -4 \\ -4 & 9 \end{pmatrix}$ **B.** $\begin{pmatrix} -1 & 2 \\ 6 & 3 \end{pmatrix}$ **C.** $\begin{pmatrix} 0 & 3 \\ 5 & -4 \end{pmatrix}$ **D.** $\begin{pmatrix} 3 & -5 \\ -5 & 6 \end{pmatrix}$

A7 What is the inverse matrix of $\begin{pmatrix} 3 & 2 \\ 2 & 1 \end{pmatrix}$

A. $\begin{pmatrix} 1 & -2 \\ -2 & 3 \end{pmatrix}$ **B.** $\begin{pmatrix} \frac{1}{3} & \frac{1}{2} \\ \frac{1}{2} & 1 \end{pmatrix}$ **C.** $\begin{pmatrix} -1 & 2 \\ 2 & -3 \end{pmatrix}$ **D.** $\begin{pmatrix} 3 & -2 \\ -2 & 1 \end{pmatrix}$

A8 What is the determinant of $\begin{pmatrix} 1 & -1 & 1 \\ 1 & 2 & 3 \\ a & b & c \end{pmatrix}$?

A. $6bc + 3ac + 3ab$ **B.** $-5a - 2b + 3c$ **C.** $a - b + c + 1$ **D.** $a - 2b + 3c$

A9 The numbers a, b and c are such that $\begin{vmatrix} a & b & c \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{vmatrix} = -2$. Let $d_1 = \begin{vmatrix} 2 & 1 & 0 \\ a & b & c \\ -1 & 1 & 1 \end{vmatrix}$, $d_2 = \begin{vmatrix} a+2 & b+1 & c \\ 2 & 1 & 0 \\ -1 & 1 & 1 \end{vmatrix}$ and $d_3 = \begin{vmatrix} a & b & 2c \\ 2 & 1 & 0 \\ -1 & 1 & 2 \end{vmatrix}$. Which of the following holds?

A. $d_1 > d_2 > d_3$ **B.** $d_2 > d_3 > d_1$ **C.** $d_3 > d_1 > d_2$ **D.** $d_3 > d_2 > d_1$

A10 The vector $\begin{pmatrix} 1 \\ -1 \end{pmatrix}$ is an eigenvector of $\begin{pmatrix} 1 & 5 \\ 6 & 2 \end{pmatrix}$. What is the corresponding eigenvalue?

A. 7 **B.** -7 **C.** 4 **D.** -4

A11 Write down the characteristic polynomial of $\begin{pmatrix} 2 & 3 \\ -5 & -7 \end{pmatrix}$.

A. $\lambda^2 - 9\lambda + 1$ **B.** $\lambda^2 - 5\lambda + 3$ **C.** $\lambda^2 + 5\lambda + 1$ **D.** $\lambda^2 - 3\lambda - 2$

A12 What is the product of the eigenvalues of $\begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}$?

A. 6 **B.** 0 **C.** -3 **D.** 3

A13 Consider the function $f(x, y) = \sin(xy^2)$. What is $\dfrac{\partial f}{\partial x}$?

A. $2y\sin(xy^2)$ **B.** $y^2\cos(xy^2)$ **C.** $\cos(2xy)$ **D.** $\cos(y^2)$

A14 What is the tangent plane to $z = x^2 - xy^2$ at the point $(1, -2, -3)$?

A. $z = x - 2y$ **B.** $z = -3$ **C.** $x - 2y - 3z = 14$ **D.** $2x - 4y + z = 7$

A15 Recall that $\cosh x = \dfrac{e^x + e^{-x}}{2}$ and $\sinh x = \dfrac{e^x - e^{-x}}{2}$. Simplify $\cosh x \cosh y + \sinh x \sinh y$.

A. $\cosh(x - y)$ **B.** e^{x+y} **C.** $\cosh(x + y)$ **D.** e^{x-y}

A16 The curve given by the quadratic formula $3x^2 - 2y^2 - 6x - 4y = 0$ is

A. a hyperbola **B.** a parabola **C.** an ellipse **D.** a circle

A17 A real symmetric 3×3 matrix has two eigenvectors $\begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix}$ and $\begin{pmatrix} 1 \\ 1 \\ 0 \end{pmatrix}$. What is the other?

A. $\begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$ **B.** $\begin{pmatrix} -1 \\ 1 \\ 1 \end{pmatrix}$ **C.** $\begin{pmatrix} 2 \\ 1 \\ 2 \end{pmatrix}$ **D.** $\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}$

A18 Let $f(x,y) = xy + x^3 - 2y^2$. The Hessian matrix at $(1,1)$ is:

A. $\begin{pmatrix} 3 & 1 \\ 1 & -2 \end{pmatrix}$ **B.** $\begin{pmatrix} 6 & 1 \\ 1 & -4 \end{pmatrix}$ **C.** $\begin{pmatrix} 3 & 0 \\ 0 & 2 \end{pmatrix}$ **D.** $\begin{pmatrix} 6 & 0 \\ 0 & -4 \end{pmatrix}$

A19 The triangle T has vertices at $(-2,1)$, $(2,1)$ and $(0,-1)$. We can compute its area with

A. $\int_{-2}^{2} \int_{y-1}^{y+1} 1 \; dx \; dy$ **B.** $\int_{-1}^{1} \int_{y-1}^{x-2} 1 \; dy \; dx$ **C.** $\int_{-1}^{1} \int_{-y-1}^{y+1} 1 \; dx \; dy$ **D.** $\int_{-2}^{2} \int_{-x-1}^{x-1} 1 \; dy \; dx$

A20 The integral $\int_0^2 \int_{-y}^{0} x \, dx \, dy$ can also be computed with

A. $\int_0^2 \int_0^x x \, dy \, dx$ **B.** $\int_0^2 \int_0^x y \, dy \, dx$ **C.** $\int_{-2}^0 \int_{-x}^2 x \, dy \, dx$ **D.** $\int_{-2}^0 \int_{-y}^0 y \, dy \, dx$

A21 What is $\int_0^1 \int_0^y x + y \, dx \, dy$?

A. $\frac{1}{2}$ **B.** $\frac{1}{3}$ **C.** $\frac{1}{4}$ **D.** $\frac{1}{5}$

A22 Write $x = uv$ and $y = u/v$. Compute $\frac{\partial(x,y)}{\partial(u,v)}$.

A. $-2u^2/v^2$ **B.** 0 **C.** $1 - u^2/v^2$ **D.** $-2u/v$

Section B

B1 Give the general solution to the system of equations:

$$\begin{aligned} x - y + z &= 2 \\ 2x + y + 2z &= 1 \\ 3x - y + 3z &= 4. \end{aligned}$$

Geometrically, what is the set of solutions? ***(3 marks)***

B2 Give explicit examples of matrices A and B where AB exists but BA does not. ***(2 marks)***

B3 (i) State the multiplication rule for determinants.

(ii) The adjoint matrix $\text{adj}(A)$ of an invertible $n \times n$ matrix A satisfies $A.\text{adj}(A) = \det A.I_n$. Express $\det(\text{adj}(A))$ in terms of $\det A$. ***(3 marks)***

B4 Let $A = \begin{pmatrix} -7 & -18 \\ 3 & 8 \end{pmatrix}$. Find a matrix M such that $M^{-1}AM$ is a diagonal matrix, and write down this diagonal matrix explicitly. ***(4 marks)***

B5 Consider the function $z = e^{xy}$. By computing its partial derivatives, give the Taylor series around $(0, 0)$ as far as the degree 2 terms.

Use the approximation you found above to predict the value of the function at $(0.1, 0.2)$. To how many decimal places is this correct? ***(4 marks)***

B6 If $y = \sinh^{-1} x$, show that

$$\frac{dy}{dx} = \frac{1}{\sqrt{1 + x^2}}.$$

Find $\int \sinh^{-1} x \, dx$. ***(4 marks)***

B7 Find the stationary points of the function $f(x, y) = x^3 + y^3 - 3x - 3y$, and determine the nature of each stationary point. ***(4 marks)***

B8 Find the area of the surface obtained by rotating the graph of $f(x) = x^3/3$ about the x-axis over the interval $[0, 1]$. ***(4 marks)***

B9 Using the substitutions $x = u$ and $y = \sqrt{u + v}$, evaluate

$$\iint_D y \cos(y^2 - x) \, dx \, dy,$$

over the region $D = \{(x, y) \mid 0 \leq x \leq \pi, \ 0 \leq y \leq \sqrt{x}\}$. ***(5 marks)***

End of Question Paper