The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2018–2019**

Vectors and Mechanics **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets. The total number of marks available is 60. All vector quantities must be underlined.

1 The points A and B have position vectors $\boldsymbol{a}$ and $\boldsymbol{b}$ respectively, where

$$\boldsymbol{a} = 3\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}, \qquad \boldsymbol{b} = 2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}.$$

Find

(i) The vector $\overrightarrow{AB}$;

(ii) The distance between A and B;

(iii) A unit vector in the direction of $\overrightarrow{AB}$;

(iv) The position vector of the mid-point M of AB;

(v) The vector equation of the line AB. ***(7 marks)***

2 You are given the vectors

$$\boldsymbol{u} = 2\boldsymbol{i} + \boldsymbol{j} - 2\boldsymbol{k}, \qquad \boldsymbol{v} = 3\boldsymbol{i} - 2\boldsymbol{j} - \boldsymbol{k}.$$

Find

(i) $\boldsymbol{u} \cdot \boldsymbol{v}$;

(ii) $\boldsymbol{u} \times \boldsymbol{v}$;

(iii) The angle between $\boldsymbol{u}$ and $\boldsymbol{v}$ in radians to an accuracy of three decimal places. ***(5 marks)***

3 Simplify the following expressions

(i) $\boldsymbol{i} + (\boldsymbol{k} \times \boldsymbol{j})$;

(ii) $[\boldsymbol{i} + (\boldsymbol{k} \times \boldsymbol{i})] \cdot \boldsymbol{j}$. ***(2 marks)***

4 Consider the plane with vector equation

$$\boldsymbol{r}.(\boldsymbol{i}-\boldsymbol{j}+\boldsymbol{k})=0$$

and the line with vector equation

$$\boldsymbol{r}=\lambda(3\boldsymbol{i}+7\boldsymbol{j}+4\boldsymbol{k}),$$

where λ is a parameter.

(i) Show that the line lies in the plane.

(ii) What is the vector equation of the projection of the line onto the yz-plane?

(5 marks)

5 Let $\boldsymbol{a}$, $\boldsymbol{b}$ and $\boldsymbol{c}$ be vectors such that $\boldsymbol{a}$ is parallel to $\boldsymbol{b}$ and $\boldsymbol{b}$ is perpendicular to $\boldsymbol{c}$. Show that $\boldsymbol{a}\times(\boldsymbol{b}\times\boldsymbol{c})$ is anti-parallel to $\boldsymbol{c}$. ***(3 marks)***

6 A particle is projected from the origin with speed V at an angle θ above the horizontal. There is no air resistance. Let H be the maximum vertical height reached by the particle above the point of projection. Show that

$$H=\frac{V^2}{2g}\sin^2\theta,$$

where g is gravitational acceleration. ***(4 marks)***

7 A car of mass 10^3 kg can be pushed on a horizontal road with an acceleration of $0.25\ \mathrm{ms}^{-2}$ by one woman. When a second woman, exerting the same force, comes to her aid, the acceleration is $0.55\ \mathrm{ms}^{-2}$. There is a constant resistance R acting horizontally on the car. Calculate R. ***(6 marks)***

8 A particle is moving under the action of a force $\boldsymbol{F}$ given, at time t (in seconds), by

$$\boldsymbol{F}=\left[t^2\boldsymbol{i}-3t\boldsymbol{j}+2e^t\boldsymbol{k}\right]N.$$

Find the impulse given to the particle in the interval $1\le t\le 2$. ***(5 marks)***

9 A particle moves on a straight line through a fixed point O so that at time t its displacement from O is x. The equation of motion of the particle is

$$\ddot{x}+\omega^2x=0$$

where ω is a positive constant. If at time $t=0$, we have $x=a$ and $\dot{x}=v$, show that

$$x(t)=a\cos(\omega t)+\frac{v}{\omega}\sin(\omega t).$$

(6 marks)

10 An archer pulls back 0.5 m on a bow which has a stiffness of 100 N/m. The arrow weighs 50 g. What is the velocity (rounded to two decimal places) of the arrow as soon as it is released? ***(6 marks)***

(Hint: Treat the bow as a type of spring)

11 A particle of mass M is thrown vertically upwards with speed u. In addition to gravity, there is a resistance to motion of $M\lambda V^2$, where V is its instantaneous speed, acting on the particle. Show that the maximum height H of the particle is given by

$$H = \frac{1}{2\lambda} \ln\left(1 + \frac{\lambda u^2}{g}\right).$$

(8 marks)

12 A body of mass 60 g moves with speed v as a conical pendulum with string length $5x$. If the radius of the circular motion is $4x$, find the string tension (assume $g = 10\ \text{ms}^{-2}$). ***(3 marks)***

End of Question Paper