The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2018–2019**

Numbers and Groups **2 hours**

Answer ***all questions***.
You should justify your answers carefully unless the question states otherwise.

1 (i) (a) Prove that $n! > 2^n$ for all $n \geq 4$.

(b) Prove, on the other hand, that $n! < 2^{n(n+1)/2}$ for all $n \geq 1$.
(5 marks)

(ii) Simplify the congruence $120x \equiv 45 \pmod{305}$ to a congruence of the form $x \equiv a \pmod{m}$. ***(5 marks)***

2 (i) (a) State what it means for a sequence $a_1, a_2, \ldots$ to *converge* to x.
(2 marks)

(b) Let a_n be given by

$$a_n = \frac{4^{n+1}}{(2^n + 1)(2^n - 1)}.$$

Arguing directly from the definition, show that a_n converges to 4.
(4 marks)

(c) Is a_n also a Cauchy sequence? Briefly explain your answer.
(1 mark)

(ii) What is the remainder when 4^{100} is divided by 13? ***(3 marks)***

3 Throughout this question, let $f : \mathbb{Z} \to \mathbb{Z}$ be the function which takes an integer n to its remainder upon division by 12, so that $0 \leq f(n) \leq 11$.

(i) Is f injective? Is it surjective? Justify your answers. ***(2 marks)***

(ii) Which of the following statements imply, for all integers n, that $f(n) = 7$? You do not have to justify your answers.

(a) n is of the form $19 + 12a$.

(b) n is of the form $7 + 6b$.

(c) n is of the form $7 + 24c$.

(d) n is of the form $3 + 4d$ and also of the form $1 + 3e$. ***(4 marks)***

(iii) For each of the following statements, state whether they are true or false, and in each case either give a brief explanation or a counterexample as appropriate.

(a) For all $n \in \mathbb{Z}$, we have $f(f(n)) = f(n)$.

(b) For all $n \in \mathbb{Z}$, we have $f(n + 1) - 1 = f(n)$.

(c) For all $m, n \in \mathbb{Z}$, we have $f(mn) = f(m)f(n)$.

(d) For all $m, n \in \mathbb{Z}$, we have $f(mn) = f(f(m)f(n))$. ***(4 marks)***

4 (i) For each of the following sets and operations, decide whether they form a group or not, and in each case either give a brief explanation or a counterexample as appropriate.

(a) $G = \mathbb{N}$ and the operation is given by $+$.

(b) $G = \mathbb{R}$ and the operation is given by $\times$.

(c) Matrices of the form $\begin{bmatrix} a & b \\ c & d \end{bmatrix}$ with $a, b, c, d \in \mathbb{R}$ such that $ad - cb \neq 0$, and the operation is given by matrix multiplication. ***(4 marks)***

(ii) Define a relation on the set of integers by $a \sim b \Leftrightarrow a - b$ is a multiple of 10. Prove that this is an equivalence relation. ***(3 marks)***

(iii) Consider the element $\alpha = (1\ 3\ 5\ 7\ 2\ 4) \in S_8$. Find the order and sign of α. ***(3 marks)***

5 Throughout this question A_4 will denote the subgroup of S_4 of elements α with $\operatorname{sgn}\alpha = 1$. The elements of A_4 are given by:

$$\{(1), (12)(34), (13)(24), (14)(23), (123), (132), (124), (142), (134), (143), (234), (243)\}.$$

(i) (a) State Lagrange's theorem.

(b) Using part (a) or otherwise, show that A_4 has no subgroup of order 5.

(c) Show that $V = \{(1), (12)(34), (13)(24), (14)(23)\} \subset A_4$ is a subgroup of order 4. ***(5 marks)***

(ii) For this part we assume that $H \subset A_4$ is a subgroup of order 6.

(a) How many distinct cosets are there of H in A_4?

(b) Let $a \in A_4$ be an element of order 3. By considering the cosets H, aH, a^2H, prove that $a \in H$.

(c) Therefore prove that no such H exists.

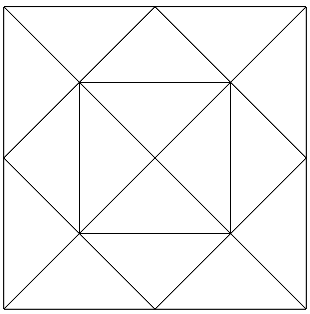
(d) Why does this result not contradict Lagrange's theorem? ***(5 marks)***

6 (i) For this question, let G be a group acting on a set X.

(a) For an element $g \in G$, define the set $\operatorname{fix}(g)$.

(b) State the orbit-counting theorem. ***(2 marks)***

(ii) A Christmas ornament, which can be turned over and rotated, is to be made by gluing together 16 triangular pieces of plastic in the following manner:



Supposing that there are 8 red and 8 green pieces, find the number of essentially different ways of making the ornament. Explain your result clearly, giving any formulas that you are using. ***(6 marks)***

(iii) Let $\mathbb{Z}[x] := \{a_0 + a_1x^1 + a_2x^2 + \cdots a_nx^n \mid a_i \in \mathbb{Z}\}$ be the set of all polynomials with integer coefficients. By using the fact that a countable union of countable sets is countable, or otherwise, prove that $\mathbb{Z}[x]$ is countable. ***(2 marks)***

End of Question Paper