The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2018–2019**

MAS156 Mathematics (Electrical and Aerospace) **3 hours**

*Attempt **ALL** the questions.*
Each question in Section A carries 3 marks,
each question in Section B carries 8 marks.
All solutions should be justified in full. Calculators should be relied upon for simple steps like basic arithmetic and plugging numbers into elementary functions.

Section A

A1 Let $f(x) = \dfrac{2x-3}{2-x} + 5$. State the domain of $f(x)$. Find $f^{-1}(x)$ and give the domain of $f^{-1}(x)$.

A2 Solve $4e^{2x+1} - e^2 = 0$ for x.

A3 Evaluate

$$\lim_{x\to 2} \frac{x^3-8}{x^2-4}$$

by using l'Hôspital's Rule or otherwise.

A4 Find the real and imaginary parts of

$$z = \frac{3+i}{1+3i}.$$

A5 For $y = \log_2 x$, find $\dfrac{dy}{dx}$.

A6 Let $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Write down its cofactor matrix (that is, a matrix made up of cofactors), adjoint matrix and inverse matrix.

A7 Find the indefinite integral $\displaystyle\int \frac{dx}{\sqrt{1+x^2}}\, dx$ by the substitution $x = \sinh u$.

A8 Find the indefinite integral $\displaystyle\int \frac{\ln x}{x}\, dx$ by using a suitable substitution.

A9 Find the value of a such that the following system of equations has a nontrivial solution (that is, a solution other than $x = y = z = 0$).

$$\begin{cases} x - y + z = 0, \\ 3x + y - 4z = 0, \\ ax - y - 2z = 0. \end{cases}$$

A10 For $f(x, y) = xy^2 \cosh(x^2 y)$, find $\dfrac{\partial f}{\partial x}$ and $\dfrac{\partial f}{\partial y}$.

A11 Let $A = \begin{pmatrix} 1 & -2 \\ 3 & 0 \end{pmatrix}$, $B = \begin{pmatrix} 4 & 4 \\ -1 & 1 \end{pmatrix}$. Find $A + 2B^T A$, where B^T denotes the transpose of matrix B.

A12 Find the general solution of

$$\frac{dy}{dx} + \frac{y}{x} = x^3.$$

Section B

B1 Let $\boldsymbol{a} = (5t^2, t, -t^3), \boldsymbol{b} = (\sin t, -\cos t, 0)$.

(i) Find $\boldsymbol{a} \times \boldsymbol{b}$.

(ii) Compute $\dfrac{d}{dt}(\boldsymbol{a} \times \boldsymbol{b})$.

B2 Let $A = \begin{pmatrix} 2 & -4 \\ -3 & 1 \end{pmatrix}$.

(i) Write down the characteristic equation and find all the eigenvalues of A.

(ii) Find the corresponding eigenvectors.

B3 Find the roots of the equation

$$z^3 - (2+i)z^2 + z - (2+i) = 0.$$

Hint: One solution is easier to spot than the others: try factorising the first two terms.

B4 Consider a function $y = f(x)$, which is defined by the following relation

$$y^x = x^y.$$

Taking logarithms of the both sides and differentiating, find an expression for $\dfrac{dy}{dx}$ in terms of x and y.

B5 Using basic row (or column) operations, show that

$$\begin{vmatrix} 1 & \cos x & \cos(x+y) \\ \cos x & 1 & \cos y \\ \cos(x+y) & \cos y & 1 \end{vmatrix} = 0,$$

for any real x and y.
Hint: You may start by noting $\cos(x+y) = \cos x \cos y - \sin x \sin y$.

B6 By using the Laplace transform, or otherwise, solve the following differential equation

$$\frac{dy}{dt} - 3y = e^{3t},$$

where $y = 1$ when $t = 0$.

B7 Answer the following two questions which are independent of each other.

(i) If $U(x,y) = \tan^{-1}\dfrac{y}{x}$, show that

$$\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} = 0$$

holds.

(ii) If $V(x,t) = \dfrac{1}{\sqrt{t}}\exp\left(\dfrac{x^2}{t}\right)$ for $t > 0$, show that

$$\frac{\partial V}{\partial t} = -\frac{1}{4}\frac{\partial^2 V}{\partial x^2}$$

holds.

B8 Consider the differential equation

$$a\frac{d^2y}{dx^2} + b\frac{dy}{dx} + cy = 0,$$

where a, b, c are constants. The auxiliary equation $\phi(k) = 0$ has roots k_1 and k_2, where $\phi(k) = ak^2 + bk + c$.

(i) Show that

$$a\frac{d^2e^{kx}}{dx^2} + b\frac{de^{kx}}{dx} + ce^{kx} = \phi(k)e^{kx}. \qquad (1)$$

Hence, show that e^{k_1x} and e^{k_2x} are two independent solutions, when $k_1 \neq k_2$.

(ii) By differentiating (1) *partially with respect to* k, that is, calculating $\frac{\partial}{\partial k}$ (1), show that

$$a\frac{d^2(xe^{kx})}{dx^2} + b\frac{d(xe^{kx})}{dx} + c(xe^{kx}) = \{\phi'(k) + \phi(k)x\}\, e^{kx}. \qquad (2)$$

(iii) For the case of repeated root $k_1 = k_2$, show how $\phi(k)$ takes a simplified form by writing it down explicitly. Hence show that the two independent solutions are given by e^{k_1x} and xe^{k_1x} using (1) and (2).

End of Question Paper

MAS140/151/152/156/161 Formula Sheet

These results may be quoted without proof unless proofs are asked for in the questions.

Trigonometry

$\sin^2 x + \cos^2 x = 1$

$1 + \tan^2 x = \sec^2 x$

$1 + \cot^2 x = \mathrm{cosec}^2 x$

$2\sin^2 x = 1 - \cos 2x$

$2\cos^2 x = 1 + \cos 2x$

$2\sin x \cos x = \sin 2x$

$\cos(x \pm y) = \cos x \cos y \mp \sin x \sin y$

$\sin(x \pm y) = \sin x \cos y \pm \cos x \sin y$

$$\tan(x \pm y) = \frac{\tan x \pm \tan y}{1 \mp \tan x \tan y}$$

$a\cos x + b\sin x = R\cos(x - \alpha)$

where $R = \sqrt{a^2 + b^2}$,

$\cos\alpha = \frac{a}{R}$ and $\sin\alpha = \frac{b}{R}$

$2\cos x \cos y = \cos(x + y) + \cos(x - y)$

$2\sin x \sin y = \cos(x - y) - \cos(x + y)$

$2\sin x \cos y = \sin(x + y) + \sin(x - y)$

$$\cos x = \frac{1 - \tan^2(x/2)}{1 + \tan^2(x/2)}$$

$$\sin x = \frac{2\tan(x/2)}{1 + \tan^2(x/2)}$$

$$\tan x = \frac{2\tan(x/2)}{1 - \tan^2(x/2)}$$

Hyperbolic Functions

$\sinh x = \frac{1}{2}(e^x - e^{-x})$

$\cosh x = \frac{1}{2}(e^x + e^{-x})$

$$\tanh x = \frac{\sinh x}{\cosh x}$$

$$\coth x = \frac{\cosh x}{\sinh x}$$

$$\mathrm{sech} x = \frac{1}{\cosh x}$$

$\cosh^2 x - \sinh^2 x = 1$

$2\cosh^2 x = 1 + \cosh 2x$

$2\sinh^2 x = \cosh 2x - 1$

$2\sinh x \cosh x = \sinh 2x$

$\mathrm{sech}^2 x = 1 - \tanh^2 x$

$\sinh^{-1} x = \ln(x + \sqrt{x^2 + 1})$, all x

$\cosh^{-1} x = \ln(x + \sqrt{x^2 - 1})$, $x \geq 1$

$$\tanh^{-1} x = \frac{1}{2}\ln\left(\frac{1 + x}{1 - x}\right), \quad |x| < 1$$

Series

Sum of an arithmetic series:

$$\frac{\text{first term} + \text{last term}}{2} \times (\text{number of terms})$$

Sum of a geometric series: $1 + x + x^2 + \ldots + x^{n-1} = \dfrac{1 - x^n}{1 - x}$

Binomial theorem: $(1+x)^n = 1 + nx + \dfrac{n(n-1)}{2!}x^2 + \ldots + \dbinom{n}{r}x^r + \ldots$

where $\dbinom{n}{r} = \dfrac{n(n-1)(n-2)\ldots(n-r+1)}{r!}$

If n is a positive integer then the series terminates and the result is true for all x, otherwise, the series is infinite and only converges for $|x| < 1$.

$$\left.\begin{aligned}
\sin x &= x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \ldots \\
\cos x &= 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \ldots \\
\sinh x &= x + \frac{x^3}{3!} + \frac{x^5}{5!} + \frac{x^7}{7!} + \ldots \\
\cosh x &= 1 + \frac{x^2}{2!} + \frac{x^4}{4!} + \frac{x^6}{6!} + \ldots \\
\exp x &= e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots
\end{aligned}\right\} \text{valid for all } x$$

$$\ln(1+x) = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \ldots \quad (-1 < x \leq 1)$$

Differentiation

Function	Derivative	Function	Derivative
$\sin x$	$\cos x$	$\cos x$	$-\sin x$
$\tan x$	$\sec^2 x$	$\cot x$	$-\text{cosec}^2 x$
$\sin^{-1} x$	$\dfrac{1}{\sqrt{1-x^2}}\,,\ \lvert x\rvert < 1$	$\cos^{-1} x$	$-\dfrac{1}{\sqrt{1-x^2}}\,,\ \lvert x\rvert < 1$
$\tan^{-1} x$	$\dfrac{1}{1+x^2}$	$\cot^{-1} x$	$-\dfrac{1}{1+x^2}$
$\sinh x$	$\cosh x$	$\cosh x$	$\sinh x$
$\tanh x$	$\dfrac{1}{\cosh^2 x}$	$\coth x$	$-\dfrac{1}{\sinh^2 x}$
$\sinh^{-1} x$	$\dfrac{1}{\sqrt{x^2+1}}$	$\cosh^{-1} x$	$\dfrac{1}{\sqrt{x^2-1}}\,,\ \lvert x\rvert > 1$
$\tanh^{-1} x$	$\dfrac{1}{1-x^2}\,,\ \lvert x\rvert < 1$		
$\coth^{-1} x$	$\dfrac{1}{1-x^2}\,,\ \lvert x\rvert > 1$		

Integration

In the following table the constants of integration have been omitted.

$$\int x^n\,dx = \frac{x^{n+1}}{n+1} \quad (n \neq -1)$$

$$\int \frac{dx}{x} = \ln|x|$$

$$\int e^x\,dx = e^x$$

$$\int a^x\,dx = \frac{a^x}{\ln a} \quad (a>0,\ a\neq 1)$$

$$\int \sin x\,dx = -\cos x$$

$$\int \cos x\,dx = \sin x$$

$$\int \sec^2 x\,dx = \tan x$$

$$\int \mathrm{cosec}^2 x\,dx = -\cot x$$

$$\int \sinh x\,dx = \cosh x$$

$$\int \cosh x\,dx = \sinh x$$

$$\int \mathrm{sech}^2 x\,dx = \tanh x$$

$$\int \mathrm{cosech}^2 x\,dx = -\coth x$$

$$\int \frac{dx}{\sqrt{a^2-x^2}} = \sin^{-1}\frac{x}{a} \quad (|x|<a)$$

$$\int \frac{dx}{a^2+x^2} = \frac{1}{a}\tan^{-1}\frac{x}{a}$$

$$\int \frac{dx}{\sqrt{x^2+a^2}} = \sinh^{-1}\frac{x}{a}$$

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \cosh^{-1}\frac{x}{a} \quad (|x|>a)$$

$$\int \frac{dx}{a^2-x^2} = \frac{1}{2a}\ln\left|\frac{a+x}{a-x}\right| \quad \left(= \frac{1}{a}\tanh^{-1}\frac{x}{a} \quad \text{if } |x|<a\right)$$

$$\int \mathrm{cosec}\,x\,dx = \ln\tan\left(\frac{x}{2}\right) \quad \text{or} \quad \ln\left(\mathrm{cosec}\,x - \cot x\right)$$

$$\int \sec x\,dx = \ln\tan\left(\frac{x}{2}+\frac{\pi}{4}\right) \quad \text{or} \quad \ln\left(\sec x + \tan x\right)$$

$$\int \mathrm{cosech}\,x\,dx = \ln\tanh\left(\frac{x}{2}\right)$$

Integration by parts

$$\int f(x)\, g'(x)\, dx = f(x)\, g(x) - \int f'(x)\, g(x)\, dx$$

Variable substitution in definite integral

If $x = \varphi(t)$ is a monotonic function in the interval $[\alpha, \beta]$ and $a = \varphi(\alpha)$, $b = \varphi(\beta)$, then

$$\int_a^b f(x)\, dx = \int_\alpha^\beta f(\varphi(t))\, \varphi'(t)\, dt$$

Variable substitution for a rational function of sin x and cos x

Let $t = \tan\left(\frac{x}{2}\right)$ then $\sin x = \dfrac{2t}{1+t^2}$, $\cos x = \dfrac{1-t^2}{1+t^2}$ and $\dfrac{dx}{dt} = \dfrac{2}{1+t^2}$.

Table of Laplace transforms

Function $f(t)$	Laplace transform $F(s)$
t^n	$\dfrac{n!}{s^{n+1}}$ (for $n = 0, 1, 2, \ldots$)
e^{at}	$\dfrac{1}{s-a}$
$\sin \omega t$	$\dfrac{\omega}{s^2+\omega^2}$
$\cos \omega t$	$\dfrac{s}{s^2+\omega^2}$
$\sinh \omega t$	$\dfrac{\omega}{s^2-\omega^2}$
$\cosh \omega t$	$\dfrac{s}{s^2-\omega^2}$
$e^{at} f(t)$	$F(s-a)$ (shift theorem)
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2F(s) - sf(0) - f'(0)$