The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2018–2019**

Algebra **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets. There is a total of 60 marks.

Please leave this exam paper on your desk
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1 (i) Let R be a ring. What does it mean for R to be commutative? ***(1 mark)***

(ii) Let S_5 be the group of permutations of the set $\{1,2,3,4,5\}$. Let $\alpha = (1\,2)(3\,4) \in S_5$. How big are the conjugacy class $\text{conj}_{S_5}(\alpha)$ and the centraliser $\text{cent}_{S_5}(\alpha)$? ***(2 marks)***

(iii) Write 91 as a product of irreducibles in the ring of Gaussian integers $\mathbb{Z}[i]$. ***(1 mark)***

(iv) Write $x^4 - 1$ as a product of irreducible elements in the rings

(a) $\mathbb{R}[x]$;

(b) $\mathbb{C}[x]$.

(2 marks)

(v) Write down the dimension of Hamilton's quaternion ring $\mathbb{H}$, as an $\mathbb{R}$-vector space. ***(1 mark)***

(vi) Write down the rank of the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 5 & 5 & 19 \end{pmatrix}$. ***(1 mark)***

(vii) Is the quotient ring $\mathbb{R}[x]/\langle x^2+1\rangle$ a field? (Yes or No.) ***(1 mark)***

(viii) Is the quotient ring $\mathbb{R}[x]/\langle x^2-1\rangle$ a field? (Yes or No.) ***(1 mark)***

(ix) Is the subset $\{e^x, e^{2x}, e^{3x}\}$ of the $\mathbb{R}$-vector space $C(\mathbb{R},\mathbb{R})$ linearly independent? (Yes or No.) ***(1 mark)***

(x) Is the subset $\{x, 1+x, 2+x\}$ of the $\mathbb{R}$-vector space $\mathbb{R}[x]$ linearly independent? (Yes or No.) ***(1 mark)***

(xi) Is the ring $\mathbb{R}[x]$ commutative? (Yes or No.) ***(1 mark)***

(xii) Is $2i$ a unit in $\mathbb{C}$? (Yes or No.) ***(1 mark)***

(xiii) In an inner product space, given a linear operator T, do orthogonal eigenvectors have to have distinct eigenvalues? (Yes or No.) ***(1 mark)***

(xiv) In an inner product space, given a linear operator T, do eigenvectors with distinct eigenvalues have to be orthogonal? (Yes or No.) ***(1 mark)***

2 Let G and H be groups, with neutral elements e_G and e_H respectively.

(i) What does it mean for a map $f : G \to H$ to be a *homomorphism*? ***(1 mark)***

(ii) What is meant by the *kernel*, $\ker f$? ***(1 mark)***

(iii) Let f be as above. Prove that $f(e_G) = e_H$, and that for any $b \in G$, $(f(b))^{-1} = f(b^{-1})$. [Hints: consider $f(e_G e_G)$ and $f(bb^{-1})$.] ***(2 marks)***

(iv) Hence show that $f(a) = f(b) \iff ab^{-1} \in \ker f$. Deduce that if $\ker f = \{e_G\}$ then f is injective. ***(4 marks)***

(v) Let G be a group and N a normal subgroup of G. Prove that the map $f : G \to G/N$ defined by $f(g) := gN \ \forall g \in G$ is a homomorphism. ***(1 mark)***

(vi) Let S_4 be the group of permutations of the set $\{1, 2, 3, 4\}$. You may assume that $V_4 := \{\mathrm{id}, (12)(34), (13)(24), (14)(23)\}$ is a normal subgroup. Using (iv), prove that the map $f : S_3 \to S_4/V_4$ given by $f(g) := gV_4$ is injective (where S_3 is viewed as a subgroup of S_4 in the natural way). Why must it be an isomorphism? ***(3 marks)***

3 Consider the matrix ring $M_2(\mathbb{R})$, with the usual addition and multiplication.

(i) Given an element $\begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $M_2(\mathbb{R})$, how do you tell whether or not it has a multiplicative inverse in $M_2(\mathbb{R})$? ***(1 mark)***

(ii) Is $B = \begin{pmatrix} 1 & 1 \\ 3 & 5 \end{pmatrix}$ invertible as an element of the subring $M_2(\mathbb{Z})$? Find an invertible element D of $M_2(\mathbb{Z})$ whose first column is $\begin{pmatrix} 31 \\ 11 \end{pmatrix}$. ***(3 marks)***

4 Let $\mathbb{F}_2 = \mathbb{Z}/2\mathbb{Z} = \{\overline{0}, \overline{1}\}$, the field with 2 elements. We define a subset

$$S := \left\{ \begin{pmatrix} a & b \\ b & a+b \end{pmatrix} \ : \ a, b \in \mathbb{F}_2 \right\}$$

of $R := M_2(\mathbb{F}_2)$.

(i) Prove that S is a subring of $M_2(\mathbb{F}_2)$. ***(4 marks)***

(ii) Write down the dimension of S as an $\mathbb{F}_2$-vector space ***(1 mark)***

(iii) Is S a field? Justify your answer. ***(2 marks)***

5 (i) Let F be a field, and let V and W be F-vector spaces. What does it mean for a map $\ell : V \to W$ to be *linear*? ***(1 mark)***

(ii) Let $\ell_1, \ell_2 : V \to W$ be linear maps. Prove that if we define $\ell_1 + \ell_2 : V \to W$ by $(\ell_1 + \ell_2)(v) := \ell_1(v) + \ell_2(v)\ \forall v \in V$, then $\ell_1 + \ell_2$ is also linear. ***(2 marks)***

6 (i) Let $V = C^\infty(\mathbb{R}, \mathbb{R})$, the $\mathbb{R}$-vector space of real-valued functions, with derivatives of all orders, of a real variable. Let $L(V)$ be the ring of linear operators on V, and consider $D \in L(V)$ defined by $D(y) := \frac{dy}{dx}$. By solving a homogeneous second-order differential equation, find a basis for the subspace $\ker(D^2 + 2D + 5)$ of V (where "5" means multiplication by 5, so $5(y) = 5y$). ***(2 marks)***

(ii) With V and D as above, and letting $\theta := D^2 + 2D + 5$, find $v \in V$ such that under the isomorphism $V/\ker(\theta) \simeq \operatorname{im}(\theta)$, we have $v + \ker(\theta) \mapsto x$. ***(2 marks)***

7 Let $\mathbb{R}[x]_{\leq 5}$ be the $\mathbb{R}$-vector space of polynomials of degree at most 5. Let $\frac{d}{dx} \in L(\mathbb{R}[x]_{\leq 5})$ be the linear operator sending a polynomial to its derivative. Let D be the matrix representing $\frac{d}{dx}$ with respect to the basis $\{1, x, x^2, x^3, x^4, x^5\}$. Write down the matrices D, D^5 and D^{99}. ***(3 marks)***

8 You may assume that $\langle f, g\rangle := \int_0^1 f(x)g(x)\,dx$ defines an inner product on the $\mathbb{R}$-vector space $C([0,1], \mathbb{R})$ of continuous real-valued functions on the interval $[0,1]$.

(i) Find $\langle 1, 1\rangle$, $\langle 1, x\rangle$ and $\langle x, x\rangle$. ***(1 mark)***

(ii) What are the length of x and the angle between 1 and x, with respect to this inner product? ***(2 marks)***

(iii) Without any further integration, find an element $f \in \operatorname{Span}\{1, x\}$ such that $\{1, f\}$ is an orthonormal basis for $\operatorname{Span}\{1, x\}$. ***(2 marks)***

9 Let V be a vector space over $\mathbb{R}$, with an inner product $\langle , \rangle$. Let $T \in L(V)$ be a linear operator (i.e. a linear map from V to V).

(i) What does it mean for T to be *self-adjoint* with respect to $\langle , \rangle$? ***(1 mark)***

(ii) Now let $V = \mathbb{R}^n$, with the standard dot product, i.e.

$$\langle \mathbf{x}, \mathbf{y} \rangle = \mathbf{x} \cdot \mathbf{y} := \mathbf{x}^t \, \mathbf{y} = \sum_{i=1}^{n} x_i y_i, \text{ for any } \mathbf{x}, \mathbf{y} \in \mathbb{R}^n.$$

Let $T \in L(\mathbb{R}^n)$ be defined by $T(\mathbf{x}) := A\mathbf{x}$, where $A \in M_n(\mathbb{R})$ is fixed. Prove that T is self-adjoint with respect to $\langle , \rangle$ if and only if A is *symmetric*. ***(2 marks)***

(iii) Is $\mathrm{ref}_{\pi/2} \in L(\mathbb{R}^2)$ self-adjoint? For what values of $\alpha \in \mathbb{R}$ is $\mathrm{rot}_\alpha \in L(\mathbb{R}^2)$ self-adjoint? Justify your answers. ***(3 marks)***

End of Question Paper