SCHOOL OF MATHEMATICS AND STATISTICS

Spring Semester
2018–2019

Algebra

2 hours 30 minutes

Attempt all the questions. The allocation of marks is shown in brackets. There is a total of 60 marks.

Please leave this exam paper on your desk
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Registration number from U-Card (9 digits)
to be completed by student

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(i) Let $R$ be a ring. What does it mean for $R$ to be commutative? (1 mark)

(ii) Let $S_5$ be the group of permutations of the set $\{1, 2, 3, 4, 5\}$. Let $\alpha = (12)(34) \in S_5$. How big are the conjugacy class $\text{conj}_{S_5}(\alpha)$ and the centraliser $\text{cent}_{S_5}(\alpha)$? (2 marks)

(iii) Write $91$ as a product of irreducibles in the ring of Gaussian integers $\mathbb{Z}[i]$. (1 mark)

(iv) Write $x^4 - 1$ as a product of irreducible elements in the rings
(a) $\mathbb{R}[x]$;
(b) $\mathbb{C}[x]$. (2 marks)

(v) Write down the dimension of Hamilton’s quaternion ring $\mathbb{H}$, as an $\mathbb{R}$-vector space. (1 mark)

(vi) Write down the rank of the matrix $\begin{pmatrix} 1 & 1 & 1 \\ 2 & 2 & 3 \\ 5 & 5 & 19 \end{pmatrix}$. (1 mark)

(vii) Is the quotient ring $\mathbb{R}[x]/\langle x^2 + 1 \rangle$ a field? (Yes or No.) (1 mark)

(viii) Is the quotient ring $\mathbb{R}[x]/\langle x^2 - 1 \rangle$ a field? (Yes or No.) (1 mark)

(ix) Is the subset $\{e^x, e^{2x}, e^{3x}\}$ of the $\mathbb{R}$-vector space $C(\mathbb{R}, \mathbb{R})$ linearly independent? (Yes or No.) (1 mark)

(x) Is the subset $\{x, 1+x, 2+x\}$ of the $\mathbb{R}$-vector space $\mathbb{R}[x]$ linearly independent? (Yes or No.) (1 mark)

(xi) Is the ring $\mathbb{R}[x]$ commutative? (Yes or No.) (1 mark)

(xii) Is $2i$ a unit in $\mathbb{C}$? (Yes or No.) (1 mark)

(xiii) In an inner product space, given a linear operator $T$, do orthogonal eigenvectors have to have distinct eigenvalues? (Yes or No.) (1 mark)

(xiv) In an inner product space, given a linear operator $T$, do eigenvectors with distinct eigenvalues have to be orthogonal? (Yes or No.) (1 mark)
2 Let $G$ and $H$ be groups, with neutral elements $e_G$ and $e_H$ respectively.
(i) What does it mean for a map $f : G \to H$ to be a homomorphism?  
(1 mark)
(ii) What is meant by the kernel, $\ker f$?  
(1 mark)
(iii) Let $f$ be as above. Prove that $f(e_G) = e_H$, and that for any $b \in G$, 
$f(b^{-1}) = f(b)^{-1}$. [Hints: consider $f(e_G e_G)$ and $f(bb^{-1})$.]  
(2 marks)
(iv) Hence show that $f(a) = f(b) \iff ab^{-1} \in \ker f$. Deduce that if $\ker f = \{e_G\}$ then $f$ is injective.  
(4 marks)
(v) Let $G$ be a group and $N$ a normal subgroup of $G$. Prove that the map 
$f : G \to G/N$ defined by $f(g) := gN \forall g \in G$ is a homomorphism.  
(1 mark)
(vi) Let $S_4$ be the group of permutations of the set $\{1, 2, 3, 4\}$. You may assume 
that $V_4 := \{\text{id}, (12)(34), (13)(24), (14)(23)\}$ is a normal subgroup. Using 
(iv), prove that the map $f : S_3 \to S_4/V_4$ given by $f(g) := gV_4$ is injective 
(where $S_3$ is viewed as a subgroup of $S_4$ in the natural way). Why must it 
be an isomorphism?  
(3 marks)

3 Consider the matrix ring $M_2(\mathbb{R})$, with the usual addition and multiplication.
(i) Given an element $egin{pmatrix} a & b \\
 c & d \end{pmatrix}$ of $M_2(\mathbb{R})$, how do you tell whether or not it has 
a multiplicative inverse in $M_2(\mathbb{R})$?  
(1 mark)
(ii) Is $B = \begin{pmatrix} 1 & 1 \\
 3 & 5 \end{pmatrix}$ invertible as an element of the subring $M_2(\mathbb{Z})$? Find an 
invertible element $D$ of $M_2(\mathbb{Z})$ whose first column is $\begin{pmatrix} 31 \\
 11 \end{pmatrix}$.  
(3 marks)

4 Let $F_2 = \mathbb{Z}/2\mathbb{Z} = \{0, 1\}$, the field with 2 elements. We define a subset 
$S := \left\{ \begin{pmatrix} a & b \\
 b & a+b \end{pmatrix} : a, b \in F_2 \right\}$ 
of $R := M_2(F_2)$.
(i) Prove that $S$ is a subring of $M_2(F_2)$.  
(4 marks)
(ii) Write down the dimension of $S$ as an $F_2$-vector space  
(1 mark)
(iii) Is $S$ a field? Justify your answer.  
(2 marks)
Let $F$ be a field, and let $V$ and $W$ be $F$-vector spaces. What does it mean for a map $\ell : V \to W$ to be linear? \hfill (1 mark)

Let $\ell_1, \ell_2 : V \to W$ be linear maps. Prove that if we define $\ell_1 + \ell_2 : V \to W$ by $(\ell_1 + \ell_2)(v) := \ell_1(v) + \ell_2(v)$ $\forall v \in V$, then $\ell_1 + \ell_2$ is also linear. \hfill (2 marks)

Let $V = C^\infty(\mathbb{R}, \mathbb{R})$, the $\mathbb{R}$-vector space of real-valued functions, with derivatives of all orders, of a real variable. Let $L(V)$ be the ring of linear operators on $V$, and consider $D \in L(V)$ defined by $D(y) := \frac{dy}{dx}$. By solving a homogeneous second-order differential equation, find a basis for the subspace $\ker(D^2 + 2D + 5)$ of $V$ (where “5” means multiplication by 5, so $5(y) = 5y$). \hfill (2 marks)

With $V$ and $D$ as above, and letting $\theta := D^2 + 2D + 5$, find $v \in V$ such that under the isomorphism $V/\ker(\theta) \simeq \mathrm{im}(\theta)$, we have $v + \ker(\theta) \mapsto x$. \hfill (2 marks)

Let $\mathbb{R}[x]_{\leq 5}$ be the $\mathbb{R}$-vector space of polynomials of degree at most 5. Let $\frac{d}{dx} \in L(\mathbb{R}[x]_{\leq 5})$ be the linear operator sending a polynomial to its derivative. Let $D$ be the matrix representing $\frac{d}{dx}$ with respect to the basis $\{1, x, x^2, x^3, x^4, x^5\}$. Write down the matrices $D, D^5$ and $D^{99}$. \hfill (3 marks)

You may assume that $\langle f, g \rangle := \int_0^1 f(x)g(x) \, dx$ defines an inner product on the $\mathbb{R}$-vector space $C([0, 1], \mathbb{R})$ of continuous real-valued functions on the interval $[0, 1]$.

(i) Find $\langle 1, 1 \rangle, \langle 1, x \rangle$ and $\langle x, x \rangle$. \hfill (1 mark)

(ii) What are the length of $x$ and the angle between 1 and $x$, with respect to this inner product? \hfill (2 marks)

(iii) Without any further integration, find an element $f \in \text{Span}\{1, x\}$ such that $\{1, f\}$ is an orthonormal basis for $\text{Span}\{1, x\}$. \hfill (2 marks)
Let $V$ be a vector space over $\mathbb{R}$, with an inner product $\langle , \rangle$. Let $T \in L(V)$ be a linear operator (i.e. a linear map from $V$ to $V$).

(i) What does it mean for $T$ to be self-adjoint with respect to $\langle , \rangle$?  \hspace{1cm} (1 mark)

(ii) Now let $V = \mathbb{R}^n$, with the standard dot product, i.e.

\[
\langle x, y \rangle = x \cdot y := x^t y = \sum_{i=1}^{n} x_i y_i, \text{ for any } x, y \in \mathbb{R}^n.
\]

Let $T \in L(\mathbb{R}^n)$ be defined by $T(x) := Ax$, where $A \in M_n(\mathbb{R})$ is fixed. Prove that $T$ is self-adjoint with respect to $\langle , \rangle$ if and only if $A$ is symmetric. \hspace{1cm} (2 marks)

(iii) Is $\text{ref}_{\pi/2} \in L(\mathbb{R}^2)$ self-adjoint? For what values of $\alpha \in \mathbb{R}$ is $\text{rot}_\alpha \in L(\mathbb{R}^2)$ self-adjoint? Justify your answers. \hspace{1cm} (3 marks)

End of Question Paper