The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2018–2019**

Analysis **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) (a) Give the definition of what it means for a non–empty set A of real numbers to be *bounded above*. **(1 mark)**

(b) Let A be a non-empty subset of the real numbers that is bounded above. Define the *supremum* of A (also known as the *least upper bound* of A). **(2 marks)**

(ii) Consider the set

$$A = \left\{ 17 - \frac{1}{n} \mid n \in \mathbb{N} \right\}.$$

For each of the following, justify your answer carefully.

(a) Does A have a minimum element? If so, what is it?

(b) Does A have a maximum element? If so, what is it?

(c) Does A have an infimum? If so, what is it?

(d) Does A have a supremum? If so, what is it?

(6 marks)

(iii) (a) Carefully write down the precise mathematical definition, using ε and N, of what it means for a sequence of real numbers (a_n) to converge to a limit $l \in \mathbb{R}$. **(1 mark)**

(b) Consider the sequence (a_n) whose nth term is $a_n = 2 - \dfrac{1}{n(n+1)}$. What is the limit of this sequence? Give a careful proof that the sequence converges to this limit, using the ε, N definition. **(7 marks)**

(iv) Define a sequence of real numbers (x_n) by

$$x_1 = \frac{3}{2} \qquad \text{and} \qquad x_{n+1} = \frac{1}{3}(x_n^2 + 2) \qquad \text{for } n > 1.$$

(a) Show that $1 \leq x_n \leq 2$ for all $n \in \mathbb{N}$.

(b) Show that $x_{n+1} - x_n = \dfrac{1}{3}(x_n - 1)(x_n - 2)$ for all $n \in \mathbb{N}$.

(c) Show that the sequence (x_n) is monotone.

(d) Explain why the sequence converges and find its limit, justifying your reasoning carefully.

(8 marks)

2 (i) Let A be a subset of $\mathbb{R}$ and let $a \in A$. Write down the precise mathematical definition of what it means for a function $f : A \to \mathbb{R}$ to be continuous at a, in terms of limits of functions. **(2 marks)**

(ii) Draw the graph of a function $g : \mathbb{R} \to \mathbb{R}$ that is continuous at all $x \in \mathbb{R} \setminus \{2\}$, but not continuous at $x = 2$. (You *do not* need to give a formula.) **(2 marks)**

(iii) (a) Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} x+1 & \text{if } x < 0, \\ 1-x & \text{if } x > 0. \end{cases}$$

Is there any continuous extension $\tilde{f} : \mathbb{R} \to \mathbb{R}$ of f ? Justify your answer. **(3 marks)**

(b) Let $g : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be defined by

$$g(x) = |x| + \frac{x}{|x|}.$$

Is there any continuous extension $\tilde{g} : \mathbb{R} \to \mathbb{R}$ of g ? Justify your answer. **(3 marks)**

(iv) (a) State the Intermediate Value Theorem. **(2 marks)**

(b) Let $f : [a,b] \to \mathbb{R}$ be a continuous function such that the image of f is $[a,b]$. Show that there exists some $c \in [a,b]$ such that $f(c) = c$.

[Hint: consider $g : [a,b] \to \mathbb{R}$ given by $g(x) = f(x) - x$.] **(7 marks)**

(v) For each of the following statements, say whether it is true or false. You should explain your answer, giving a suitable counterexample where relevant.

(a) Let $f : [a,b] \to \mathbb{R}$ be a continuous function which is not constant. The image of f is $[m, M]$ for some $m < M$. **(2 marks)**

(b) Let $f : [0,1] \to \mathbb{R}$ be a bounded function. Then f has a maximum value. **(2 marks)**

(c) Let $f : (0,1) \to \mathbb{R}$ be a continuous function. Then f is bounded. **(2 marks)**

3 (i) Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 2x - 1 & \text{if } x \leq 1, \\ x^2 & \text{if } x > 1. \end{cases}$$

For each of the following statements, say whether it is true or false, carefully justifying your answer. You may use standard facts about polynomials.

(a) The function f is continuous at every $x \in \mathbb{R}$. **(3 marks)**

(b) The function f is differentiable at every $x \in \mathbb{R}$. **(4 marks)**

(c) The derivative function f' has domain $\mathbb{R}$ and is continuous at every $x \in \mathbb{R}$. **(3 marks)**

(d) The derivative function f' has domain $\mathbb{R}$ and is differentiable at every $x \in \mathbb{R}$. **(3 marks)**

(ii) (a) State the Mean Value Theorem. **(2 marks)**

(b) Let $0 < a < b$ and let $f : [a, b] \to \mathbb{R}$ be a function that is continuous on $[a, b]$ and differentiable on (a, b). Show, directly using the Mean Value Theorem, that there exists $c \in (a, b)$ such that

$$\frac{f'(c)}{2c} = \frac{f(b) - f(a)}{b^2 - a^2}.$$

[Hint: let $\alpha = \dfrac{f(b) - f(a)}{b^2 - a^2}$ and consider $h : [a, b] \to \mathbb{R}$ given by $h(x) = f(x) - \alpha x^2$.] **(6 marks)**

(iii) Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series, where $a_n \in \mathbb{R}$ for $n \geq 0$.

(a) Explain what it means to say that this power series has radius of convergence R, with $R > 0$. **(1 mark)**

(b) Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{n^4}{5n + 7} x^n.$$

(3 marks)

4 (i) (a) Let $a, b \in \mathbb{R}$, with $a < b$, and let $A \subseteq [a, b]$. We write $\mathbf{1}_A$ for the *indicator function* of A. Thus $\mathbf{1}_A\colon [a, b] \to \mathbb{R}$ is defined by

$$\mathbf{1}_A(t) = \begin{cases} 1 & \text{if } t \in A, \\ 0 & \text{if } t \notin A. \end{cases}$$

Let $s : [a, b] \to \mathbb{R}$ be a step function given by $s(t) = \sum_{i=1}^{n} \alpha_i \mathbf{1}_{[x_i, x_{i+1}]}(t)$, where $[x_i, x_{i+1}] \subseteq [a, b]$ and $\alpha_i \in \mathbb{R}$. Explain how the integral of this step function is defined. **(2 marks)**

(b) Let $f : [a, b] \to \mathbb{R}$ be a bounded function. Define the *lower integral*, $L\int_a^b f(t)\,dt$, and the *upper integral*, $U\int_a^b f(t)\,dt$. Say what it means for the function to be *Riemann integrable* and define the Riemann integral of f in that case. **(4 marks)**

(c) Let $f : [0, 4] \to \mathbb{R}$ be a monotonic decreasing continuous function with $f(0) = 7$, $f(1) = 6$, $f(2) = 5$, $f(3) = 4$ and $f(4) = 3$. Show carefully, using the definitions from parts (a) and (b), that

$$18 \leq \int_0^4 f(t)\,dt \leq 22.$$

(8 marks)

(ii) For $n \geq 1$, define a function $f_n : [0, 1] \to \mathbb{R}$ by

$$f_n(t) = \begin{cases} 1, & \text{if } 0 \leq t \leq 1 - \dfrac{1}{2^n}, \\ 2^n(1-t), & \text{if } 1 - \dfrac{1}{2^n} < t \leq 1. \end{cases}$$

You may assume that the function f_n is continuous for each $n \geq 1$.

(a) Sketch the graphs of f_1, f_2 and f_3. **(3 marks)**

(b) Show that the sequence (f_n) converges pointwise, and say what the limit function f is. **(4 marks)**

(c) Does the sequence (f_n) converge uniformly? Justify your answer. **(4 marks)**

End of Question Paper