Attempt all the questions. The allocation of marks is shown in brackets.
1. (i) (a) Give the definition of what it means for a non-empty set $A$ of real numbers to be bounded above. 

(b) Let $A$ be a non-empty subset of the real numbers that is bounded above. Define the supremum of $A$ (also known as the least upper bound of $A$). 

(ii) Consider the set 

$$A = \left\{ 17 - \frac{1}{n} \mid n \in \mathbb{N} \right\}.$$ 

For each of the following, justify your answer carefully.

(a) Does $A$ have a minimum element? If so, what is it?
(b) Does $A$ have a maximum element? If so, what is it?
(c) Does $A$ have an infimum? If so, what is it?
(d) Does $A$ have a supremum? If so, what is it?

(iii) (a) Carefully write down the precise mathematical definition, using $\varepsilon$ and $N$, of what it means for a sequence of real numbers $(a_n)$ to converge to a limit $l \in \mathbb{R}$. 

(b) Consider the sequence $(a_n)$ whose $n$th term is $a_n = 2 - \frac{1}{n(n+1)}$. What is the limit of this sequence? Give a careful proof that the sequence converges to this limit, using the $\varepsilon, N$ definition.

(iv) Define a sequence of real numbers $(x_n)$ by 

$$x_1 = \frac{3}{2} \quad \text{and} \quad x_{n+1} = \frac{1}{3}(x_n^2 + 2) \quad \text{for } n > 1.$$ 

(a) Show that $1 \leq x_n \leq 2$ for all $n \in \mathbb{N}$.
(b) Show that $x_{n+1} - x_n = \frac{1}{3}(x_n - 1)(x_n - 2)$ for all $n \in \mathbb{N}$.
(c) Show that the sequence $(x_n)$ is monotone.
(d) Explain why the sequence converges and find its limit, justifying your reasoning carefully.

Continued
(i) Let $A$ be a subset of $\mathbb{R}$ and let $a \in A$. Write down the precise mathematical definition of what it means for a function $f : A \to \mathbb{R}$ to be continuous at $a$, in terms of limits of functions. (2 marks)

(ii) Draw the graph of a function $g : \mathbb{R} \to \mathbb{R}$ that is continuous at all $x \in \mathbb{R} \setminus \{2\}$, but not continuous at $x = 2$. (You do not need to give a formula.) (2 marks)

(iii) (a) Let $f : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be defined by

$$f(x) = \begin{cases} 
  x + 1 & \text{if } x < 0, \\
  1 - x & \text{if } x > 0.
\end{cases}$$

Is there any continuous extension $\tilde{f} : \mathbb{R} \to \mathbb{R}$ of $f$? Justify your answer. (3 marks)

(b) Let $g : \mathbb{R} \setminus \{0\} \to \mathbb{R}$ be defined by

$$g(x) = |x| + \frac{x}{|x|}.$$ 

Is there any continuous extension $\tilde{g} : \mathbb{R} \to \mathbb{R}$ of $g$? Justify your answer. (3 marks)

(iv) (a) State the Intermediate Value Theorem. (2 marks)

(b) Let $f : [a, b] \to \mathbb{R}$ be a continuous function such that the image of $f$ is $[a, b]$. Show that there exists some $c \in [a, b]$ such that $f(c) = c$.

[Hint: consider $g : [a, b] \to \mathbb{R}$ given by $g(x) = f(x) - x$.] (7 marks)

(v) For each of the following statements, say whether it is true or false. You should explain your answer, giving a suitable counterexample where relevant.

(a) Let $f : [a, b] \to \mathbb{R}$ be a continuous function which is not constant. The image of $f$ is $[m, M]$ for some $m < M$. (2 marks)

(b) Let $f : [0, 1] \to \mathbb{R}$ be a bounded function. Then $f$ has a maximum value. (2 marks)

(c) Let $f : (0, 1) \to \mathbb{R}$ be a continuous function. Then $f$ is bounded. (2 marks)
3 (i) Consider the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) = \begin{cases} 
2x - 1 & \text{if } x \leq 1, \\
x^2 & \text{if } x > 1.
\end{cases}$$

For each of the following statements, say whether it is true or false, carefully justifying your answer. You may use standard facts about polynomials.

(a) The function $f$ is continuous at every $x \in \mathbb{R}$. \hspace{1cm} (3 marks)

(b) The function $f$ is differentiable at every $x \in \mathbb{R}$. \hspace{1cm} (4 marks)

(c) The derivative function $f'$ has domain $\mathbb{R}$ and is continuous at every $x \in \mathbb{R}$. \hspace{1cm} (3 marks)

(d) The derivative function $f'$ has domain $\mathbb{R}$ and is differentiable at every $x \in \mathbb{R}$. \hspace{1cm} (3 marks)

(ii) (a) State the Mean Value Theorem. \hspace{1cm} (2 marks)

(b) Let $0 < a < b$ and let $f : [a, b] \to \mathbb{R}$ be a function that is continuous on $[a, b]$ and differentiable on $(a, b)$. Show, directly using the Mean Value Theorem, that there exists $c \in (a, b)$ such that

$$\frac{f'(c)}{2c} = \frac{f(b) - f(a)}{b^2 - a^2}.$$

[Hint: let $\alpha = \frac{f(b) - f(a)}{b^2 - a^2}$ and consider $h : [a, b] \to \mathbb{R}$ given by $h(x) = f(x) - \alpha x^2$.] \hspace{1cm} (6 marks)

(iii) Let $\sum_{n=0}^{\infty} a_n x^n$ be a power series, where $a_n \in \mathbb{R}$ for $n \geq 0$.

(a) Explain what it means to say that this power series has radius of convergence $R$, with $R > 0$. \hspace{1cm} (1 mark)

(b) Find the radius of convergence of the power series

$$\sum_{n=0}^{\infty} \frac{n^4}{5n+7} x^n.$$ \hspace{1cm} (3 marks)
4 (i) (a) Let \( a, b \in \mathbb{R} \), with \( a < b \), and let \( A \subseteq [a, b] \). We write \( 1_A \) for the indicator function of \( A \). Thus \( 1_A : [a, b] \to \mathbb{R} \) is defined by

\[
1_A(t) = \begin{cases} 
1 & \text{if } t \in A, \\
0 & \text{if } t \notin A.
\end{cases}
\]

Let \( s : [a, b] \to \mathbb{R} \) be a step function given by \( s(t) = \sum_{i=1}^{n} \alpha_i 1_{[x_i, x_{i+1}]}(t) \), where \([x_i, x_{i+1}] \subseteq [a, b]\) and \( \alpha_i \in \mathbb{R} \). Explain how the integral of this step function is defined. \(\text{(2 marks)}\)

(b) Let \( f : [a, b] \to \mathbb{R} \) be a bounded function. Define the lower integral, \( L \int_a^b f(t) \, dt \), and the upper integral, \( U \int_a^b f(t) \, dt \). Say what it means for the function to be Riemann integrable and define the Riemann integral of \( f \) in that case. \(\text{(4 marks)}\)

(c) Let \( f : [0, 4] \to \mathbb{R} \) be a monotonic decreasing continuous function with \( f(0) = 7, f(1) = 6, f(2) = 5, f(3) = 4 \) and \( f(4) = 3 \). Show carefully, using the definitions from parts (a) and (b), that \( 18 \leq \int_0^4 f(t) \, dt \leq 22 \). \(\text{(8 marks)}\)

(ii) For \( n \geq 1 \), define a function \( f_n : [0, 1] \to \mathbb{R} \) by

\[
f_n(t) = \begin{cases} 
1, & \text{if } 0 \leq t \leq 1 - \frac{1}{2^n}, \\
2^n(1 - t), & \text{if } 1 - \frac{1}{2^n} < t \leq 1.
\end{cases}
\]

You may assume that the function \( f_n \) is continuous for each \( n \geq 1 \).

(a) Sketch the graphs of \( f_1, f_2 \) and \( f_3 \). \(\text{(3 marks)}\)

(b) Show that the sequence \( (f_n) \) converges pointwise, and say what the limit function \( f \) is. \(\text{(4 marks)}\)

(c) Does the sequence \( (f_n) \) converge uniformly? Justify your answer. \(\text{(4 marks)}\)

End of Question Paper