The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS

Spring Semester 2018–2019

Differential Equations

2 hours 30 minutes

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) Sketch the phase line for the following ordinary differential equation (ODE) for $-2 \leq u \leq 3$:

$$\frac{du}{dx} = (u-2)(u+1)u.$$

State all the equilibrium points, and say which are stable and which are unstable. ***(3 marks)***

(ii) Consider the following system of ODEs

$$\frac{du}{dx} = -3u + 2v,$$
$$\frac{dv}{dx} = au - 2v,$$

where a is a real number. Find the range of values of a such that the origin is (1) a saddle, (2) a node. ***(3 marks)***

Explain whether it is possible to find a value of a such that the origin is an unstable spiral. ***(1 mark)***

(iii) Consider the following system of ODEs

$$\frac{dx}{dt} = 2y(2 + x - y),$$
$$\frac{dy}{dt} = x(1 + y - x).$$

Find the equilibrium points. Determine their nature (e.g. spiral, node, centre, etc) and stability (i.e. stable or unstable). ***(8 marks)***

Sketch the nullclines. ***(2 marks)***

On a **separate diagram**, sketch the phase portrait for the system. Include sufficiently many trajectories such that the long-term behaviour of the system from any starting-point is qualitatively clear. ***(8 marks)***

2 (i) Consider the following ODE

$$x^2y'' - 2xy' + (2 + \mu^2x^2)y = 0, \tag{1}$$

where $\mu > 0$ is a positive real number. Show that the normal form for Equation (1) is given by

$$u'' + \mu^2 u = 0, \tag{2}$$

where $y(x) = u(x)v(x)$, and $v(x)$ is the solution of a first order ODE.

(6 marks)

Hence find the general solution $y(x)$ of Equation (1). ***(1 mark)***

Given boundary conditions $y'(0) = 0$ and $y(\pi) = 0$, find the values of μ for which $y(x)$ is non-trivial. ***(4 marks)***

(ii) Consider the following ODE

$$x^2y'' + 3xy' + y = 0. \tag{3}$$

Show that $x = 0$ is a regular singular point of Equation (3). ***(2 marks)***

It is known that Equation (3) has only one Frobenius series solution of the form

$$y(x) = \sum_{n=0}^{\infty} a_n x^{n+\alpha}, \tag{4}$$

where $a_0 \neq 0$. Show that $\alpha = -1$. Hence find $y(x)$. ***(7 marks)***

Use reduction of order to find another (linearly independent) solution for Equation (3). ***(5 marks)***

3 (i) Let $u(x,t) = F(x)G(t)$ be a separable solution of the first order partial differential equation (PDE)

$$x^2 \frac{\partial u}{\partial x} - \frac{\partial u}{\partial t} = 0. \tag{5}$$

Show that the functions $F(x)$ and $G(t)$ satisfy the following ordinary differential equations

$$\frac{dF}{dx} = \frac{\alpha}{x^2} F(x), \qquad \frac{dG}{dt} = \alpha G(t),$$

where α is an arbitrary constant. ***(3 marks)***

Find the general solutions for $F(x)$ and $G(t)$, and hence show that the most general separable solution of the PDE (5) is

$$u(x,t) = K \exp\left[\alpha\left(t - \frac{1}{x}\right)\right],$$

where K is an arbitrary constant. ***(5 marks)***

Show that there exists a separable solution of the PDE (5) subject to the initial condition

$$u(x,0) = \exp\left(-\frac{2}{x}\right), \qquad x > 0. \tag{6}$$

(2 marks)

Show that there does not exist a separable solution of the PDE (5) subject to the initial condition

$$u(x,0) = \frac{1}{x^2}, \qquad x > 0. \tag{7}$$

(2 marks)

(ii) Find and sketch the characteristics of the PDE (5) in the region $x > 0$. ***(3 marks)***

Hence show that the general solution of the PDE (5) is

$$u(x,t) = w\left(t - \frac{1}{x}\right)$$

where w is an arbitrary function. ***(5 marks)***

Find w if $u(x,t)$ is the solution of the PDE (5) subject to the initial condition (6). ***(2 marks)***

Find the solution of the PDE (5) subject to the initial condition (7) and comment on your result compared with that in part (i). ***(3 marks)***

4 (i) Show that the function

$$u(x,t) = \sin(\beta x)\cos(3\beta t),$$

where β is a real constant, is a *separable* solution of the PDE

$$\frac{\partial^2 u}{\partial t^2} = 9\frac{\partial^2 u}{\partial x^2} \tag{8}$$

for all values of β. ***(3 marks)***

Show that $u(0,t) = 0$ for all values of β and find the set of values of β for which $u(x,t)$ also satisfies the boundary condition $u(\pi,t) = 0$. ***(3 marks)***

Hence find the solution of the PDE (8) subject to the boundary conditions $u(0,t) = 0 = u(\pi,t)$ and the initial conditions

$$u(x,0) = \sin(x) + 2\sin(3x), \qquad \frac{\partial u}{\partial t}(x,0) = 0. \tag{9}$$

(3 marks)

(ii) Show that the PDE (8) is hyperbolic for all x and t. ***(1 mark)***

Find the characteristic equations of the PDE (8). ***(3 marks)***

Hence show that suitable characteristic coordinates for the PDE are

$$\xi = x - 3t, \qquad \eta = x + 3t. \tag{10}$$

(3 marks)

Under the change of variables $(x,t) \to (\xi(x,t), \eta(x,t))$, with $\xi(x,t)$ and $\eta(x,t)$ the characteristic coordinates given above, show that the transformed PDE for $w(\xi,\eta) = u(x,t)$ is

$$\frac{\partial^2 w}{\partial \xi \partial \eta} = 0.$$

(3 marks)

Hence show that the general solution of the PDE (8) is

$$u(x,t) = f(x - 3t) + g(x + 3t)$$

where f and g are arbitrary functions. ***(3 marks)***

Find the functions f and g if $u(x,t)$ also satisfies the boundary conditions $u(0,t) = 0 = u(\pi,t)$ and the initial conditions (9). ***(3 marks)***

End of Question Paper