The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2018–2019**

Mathematics (Computational and Numerical Methods) **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) Sketch the functions $f(x) = \sin x$ and $g(x) = \dfrac{1}{x}$ on the same graph for $-2\pi \le x \le 2\pi$. From this sketch deduce how many times

$$f(x) = g(x) \qquad (1)$$

for $-2\pi \le x \le 2\pi$.

From your sketch choose a suitable starting value of x_0 for the Newton-Raphson method to find the smallest positive value of x for which equation (1) is satisfied. Repeat the Newton-Raphson method until the value of x is found to an accuracy of three decimal places. ***(10 marks)***

(ii) Use the Gauss-Seidel iteration method to find an approximate solution to the following system of equations

$$\begin{aligned} 2x_1 + 10x_2 + 3x_3 &= 5 \\ 3x_1 + 4x_2 + 11x_3 &= 7 \\ 9x_1 + x_2 + x_3 &= 3. \end{aligned}$$

If necessary, first rearrange these equations to ensure convergence. Then starting with the initial column vector $\mathbf{x} = [0, 0, 0]^T$ compute three successive iterations, giving your final answer accurate to four decimal places. ***(7 marks)***

(iii) Using the following matrix

$$(A - 5I)^{-1} = \begin{pmatrix} 1 & 1 & -1 \\ 1 & 2 & 1 \\ 2 & -1 & 1 \end{pmatrix},$$

find the eigenvalue of matrix A closest to the value of 5 by using three iterations of the power method starting with the eigenvector $\boldsymbol{z} = [1, 1, 1]^T$. Give your final answer correct to two decimal places. ***(8 marks)***

2 (i) Consider the three points (x_0, y_0), (x_1, y_1) and (x_2, y_2), where $x_1 = x_0 + h$ and $x_2 = x_0 + 2h$ and $h > 0$ is a constant step size. From the quadratic interpolation between these three points derive the forward difference approximation for $P_2'(x_0)$, the central difference approximation for $P_2'(x_1)$ and the backward difference approximation for $P_2'(x_2)$.

Hint: The Lagrange interpolation polynomial of least degree which passes through $(n+1)$ points (x_i, y_i), $i = 0, 1, 2, \ldots, n$ is

$$P_n(x) = \sum_{i=0}^{n} L_i(x) y_i$$

where

$$L_i(x) = \frac{(x - x_0)(x - x_1)\ldots(x - x_{i-1})(x - x_{i+1})\ldots(x - x_n)}{(x_i - x_0)(x_i - x_1)\ldots(x_i - x_{i-1})(x_i - x_{i+1})\ldots(x_i - x_n)}$$

and $y_i = y(x_i)$.

(8 marks)

(ii) A polynomial of degree n can be expressed by the following sum,

$$P_n(x) = \sum_{j=0}^{n} a_j x^j.$$

In the least squares sense, a unique polynomial of degree n can be fitted to data points $(x_i, f(x_i))$, where $i = 0, 1, 2, \ldots, m$ and $m \geq n$. Assuming that the x_i values are free of errors, derive the normal equations

$$\sum_{i=0}^{m} \left(\sum_{j=0}^{n} a_j x_i^{j+k} \right) = \sum_{i=0}^{m} x_i^k f_i, \qquad k = 0, 1, 2, \ldots, n.$$

(5 marks)

(iii) Assume the attenuation of sound in air is proportional to a power of the frequency. Using a least squares fit approximation find the power law associated with the following data, assuming the frequency measurements are error free.

Frequency (kHz)	1	2	4	8	16	32
Attenuation (dB/km)	14	45	110	180	230	360

In your calculation give the final least squares fit parameters correct to an accuracy of two decimal places. ***(12 marks)***

3 (i) Use the data points (x_0, y_0) and (x_1, y_1) to derive the Trapezoidal Rule assuming $x_1 = x_0 + h$, where $h > 0$ is a constant. ***(4 marks)***

(ii) Using the Composite Trapezoidal Rule evaluate

$$\frac{1}{9}\int_0^9 \exp\left(-\frac{x^2}{180}\right)\, dx$$

to an accuracy of $\epsilon = 10^{-3}$. Give your final answer to an accuracy of three decimal places.

Hint: If a function $y(x)$ has two continuous derivatives on an interval (a, b) and this interval is divided into n subintervals, where n is a positive integer, then the error bound for the Composite Trapezoidal Rule is given by

$$\left|E_n^S\right| \leq \frac{h^2}{12}(b-a)K,$$

where

$$h = \frac{b-a}{n}$$

and

$$K = \max_{a \leq x \leq b}\left|\frac{d^2y(x)}{dx^2}\right|.$$

(13 marks)

(iii) For function $y(x)$, derive the first 4 non-zero terms of the Taylor series solution to the ordinary differential equation,

$$y' - y^3 - x^2 = -4,$$

subject to the initial condition $y(1) = 1$.

Hint: The Taylor series for a function $y(x)$ around a point $x = x_0$ is given by

$$y(x) = y(x_0) + \frac{(x-x_0)}{1!}y'(x_0) + \frac{(x-x_0)^2}{2!}y''(x_0) + \frac{(x-x_0)^3}{3!}y'''(x_0) + \ldots$$

(8 marks)

4 (i) The first and second order derivatives of the function $y(x)$ can be approximated in finite difference form by

$$y'_n = \frac{y_{n+1} - y_{n-1}}{2h} \quad \text{and} \quad y''_n = \frac{y_{n+1} - 2y_n + y_{n-1}}{h^2},$$

where y_n and $y_{n\pm1}$ denote $y(x_n)$ and $y(x_n \pm h)$, respectively. Use these relations with $h = 0.25$ to show that the ordinary differential equation

$$\frac{d^2y}{dx^2} + \frac{1}{x}\frac{dy}{dx} + \frac{1}{x^2}y = \cos x$$

with boundary conditions $y(1) = 0$ and $y'(1.75) = 1$ may be approximated by a system of linear algebraic equations in the form $\mathbf{Ay} = \mathbf{b}$. Determine the elements of matrix $\mathbf{A}$ and column vector $\mathbf{b}$ to an accuracy of four decimal places. **Do not attempt to solve these equations**.

(10 marks)

(ii) A company manufactures two types of product, P_1 and P_2, which require 2 types of raw materials, A and B. Each unit of P_1 uses 10 kilograms of A and 40 kilograms of B, while each unit of P_2 uses 10 kilograms of A and 10 kilograms of B. The raw material available each day is 60 kilograms of A and 120 kilograms of B. The company cannot produce more than 5 units of P_2 per day and it is expected that at least 1 unit of P_2 will be sold for every 2 units of P_1. The net profit per unit of P_1 is £ 2000 and of P_2 is £ 1000.

Formulate this into a linear programming problem and use graphical methods to determine the maximum possible daily profit. On the graph, clearly show the feasibility region and the line of constant revenue through the point of maximum daily profit.

(15 marks)

End of Question Paper