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The University Of Sheffield.

MAS261

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2018–2019**

Further General Engineering Mathematics **3 hours**

This paper has 100 marks. Answer ALL six questions. The questions are weighted differently. The allocation of marks is shown in brackets.

Please leave this exam paper on your desk
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to be completed by student

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1 (i) Find and classify *all* the critical points of the function

$$g(x, y) = 2x^5 - y^5 - 10x + 5y$$

(8 marks)

(ii) The vector field $\mathbf{H} : \mathbb{R}^3 \to \mathbb{R}^3$ is defined to be ∇h, where h is the scalar field given by

$$h(x, y, z) \ := \ x^3 + y^3 + z^3 + 3y^2 z + 3z^2 y,$$

(a) Calculate the divergence $\nabla \cdot \mathbf{H}$. **(2 marks)**

(b) Calculate the curl $\nabla \times \mathbf{H}$. **(2 marks)**

(c) Calculate the Laplacian $\nabla^2 \mathbf{H}$. **(2 marks)**

(iii) Let $f(x, y)$ be a function of two variables, and let $x = r\cos(\theta)$, $y = r\sin(\theta)$. Show that

$$\frac{\partial f}{\partial r} \ = \ \frac{\partial f}{\partial x}\cos(\theta) + \frac{\partial f}{\partial y}\sin(\theta).$$

Hence or otherwise, find an expression for $\dfrac{\partial^2 f}{\partial r^2}$ in terms of r and θ when $\dfrac{\partial^2 f}{\partial x^2} = \dfrac{\partial^2 f}{\partial x \partial y} = \dfrac{\partial^2 f}{\partial x^2} = 1$. **(4 marks)**

2 (i) Let $f : [0, \pi] \to \mathbb{R}$ be defined by $f(t) = t$.

(a) Find the Fourier cosine series of $f(t)$. **(11 marks)**

(b) Sketch the graph of the Fourier cosine series of $f(t)$ over the interval $[-2\pi, 2\pi]$. **(4 marks)**

(ii) Find an exact value for the series

$$\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2}.$$

You may use the fact that the Fourier series of the 2π periodic function defined by $g(x) = x - x^2$ on $[-\pi, \pi]$ is given by

$$S[g](x) = -\frac{\pi^2}{3} + \sum_{n=1}^{\infty} \left(\frac{4(-1)^{n+1}}{n^2}\cos(nx) + \frac{2(-1)^{n+1}}{n}\sin(nx) \right).$$

(4 marks)

3 (i) The function $y(t)$ satisfies the differential equation

$$y''(t) - 16y(t) \;=\; \delta(t-2)$$

subject to the initial conditions $y(0) = 1$, $y'(0) = -4$.

(a) Show that the Laplace transform $\mathcal{L}\{y\} = Y(s)$ of $y(t)$ is given by

$$Y(s) \;=\; \left(\frac{1}{s+4}\right)\left(1 + \frac{e^{-2s}}{s-4}\right)$$

(7 marks)

(b) Determine $y(t)$ for $t > 0$. **(5 marks)**

(ii) Consider the function $f(t)$ given by

$$f(t) \;=\; H(t)e^{-t} + H(-t)e^{t}$$

where $H(t)$ is the Heaviside function.

(a) Use direct integration to show that the Fourier transform $\mathcal{F}\{f(t)\} = F(\omega)$ of the function $f(t)$ is given by

$$F(\omega) = \frac{2}{1+\omega^2}.$$

(6 marks)

(b) Sketch the graph of the function $g(t)$ which has Fourier transform

$$G(\omega) = e^{-2i\omega}F(\omega)$$

(4 marks)

4 (i) Show that $12^n - 1$ is divisible by 11 for all positive integers n. **(3 marks)**

(ii) A sequence (a_n) is defined by the recurrence relation

$$a_{n+2} \;=\; 7a_{n+1} - 12a_n$$

with initial conditions $a_0 = 4$ and $a_1 = 13$. Write down the terms a_2, a_3, a_4 (answers with no working will not be given marks). **(3 marks)**

(iii) Let A and B be subsets of a set X. Determine by means a proof or a counterexample whether

$$A^c \cap B^c \;=\; (A \cup B)^c\,.$$

(3 marks)

5 (i) A region D is bounded by the curves described by the equations $y = x^2$ and $y = -x^2 + 2x + 4$.

(a) Express $\iint_D f(x,y)\, dA$ as an iterated integral integrating first with respect to y. You are not expected to evaluate the integral.
(4 marks)

(b) Express $\iint_D f(x,y)\, dA$ as an iterated integral integrating first with respect to x. You are not expected to evaluate the integral.
(4 marks)

(ii) A spherical shell $S := \{(x,y,z) \mid 81 \le x^2 + y^2 + z^2 \le 100\}$ has density

$$D(x,y,z) \;=\; \frac{1}{\sqrt{x^2+y^2+z^2}}$$

at (x,y,z). Calculate the mass of the S. **(8 marks)**

6 (i) Find the solution $f(x,t)$ of the partial differential equation

$$\frac{\partial f}{\partial x} \;=\; te^{-t}$$

subject to the boundary condition $f(0,t) = e^{-t^2}$. **(4 marks)**

(ii) (a) Use D'Alembert's solution to find the solution $u(x,t)$ to the wave equation

$$\frac{\partial^2 u}{\partial t^2} \;=\; \frac{\partial^2 u}{\partial x^2}$$

subject to the initial conditions

$$u(x,0) = 0, \qquad \frac{\partial u}{\partial t}(x,0) = \frac{4x}{(1+x^2)^2}.$$

(8 marks)

(b) Sketch the solution $u(x,t)$ at time $t = 2$. **(4 marks)**

End of Question Paper

MAS261 FORMULA SHEET

Functions:

- The Heaviside function $H(t)$ is defined by

$$H(t) \ := \ \begin{cases} 1 & \text{if } t \geq 0; \\ 0 & \text{if } t < 0. \end{cases}$$

- The rectangular function $\mathrm{rect}_T(t)$ is defined by

$$\mathrm{rect}_T(t) \ := \ \begin{cases} \frac{1}{T} & \text{if } -\frac{T}{2} \leq t \leq \frac{T}{2}; \\ 0 & \text{otherwise.} \end{cases}$$

- The sinc function $\mathrm{sinc}(t)$ is defined by

$$\mathrm{sinc}(t) \ := \ \begin{cases} \frac{\sin(t)}{t} & \text{if } t \neq 0; \\ 1 & \text{if } t = 0. \end{cases}$$

- The delta function $\delta(t)$ is defined by the property that

$$\int_{-\infty}^{\infty} \delta(t) f(t) \ dt \ = \ f(0) \quad \text{for all functions } f(t).$$

- The convolution $f * g(t)$ of two functions $f(t)$ and $g(t)$ is defined by

$$f * g(t) \ := \ \int_{-\infty}^{\infty} f(t-s) g(s) \ ds.$$

Fourier series:

The Fourier series of a periodic function $f(t)$ with fundamental period T is given by

$$S[f] = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big(a_n \cos(\omega_n t) + b_n \sin(\omega_n t) \Big)$$

where

$$\omega_n = \frac{2\pi n}{T}, \qquad a_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos(\omega_n t) dt, \qquad b_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin(\omega_n t) dt.$$

The exponential form of the Fourier series is

$$S[f] \ = \ \sum_{n=-\infty}^{\infty} c_n e^{i\omega_n t}$$

where

$$\omega_n \ = \ \frac{2\pi n}{T}, \qquad c_n = \frac{1}{T} \int_{-T/2}^{T/2} f(t) e^{-i\omega_n t} dt$$

Laplace transform:

- The Laplace transform of a function $f(t)$ is given by

$$\mathcal{L}\{f(t)\}(s) := \int_0^\infty e^{-st} f(t)\, dt.$$

Properties of the Laplace transform: $\mathcal{L}\{f(t)\} = F(s)$ in the following table.

$\mathcal{L}\{af(t)+bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$	linearity
$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$	differentiation w.r.t. t
$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$	second differentiation w.r.t. t
$\mathcal{L}\{e^{-kt}f(t)\} = F(k+s)$	frequency shift
$\mathcal{L}\{f(t-a)H(t-a)\} = e^{-as}F(s) \qquad (\text{for } a>0)$	time shift
$\mathcal{L}\{f(at)\} = \frac{1}{a}F(\frac{s}{a}) \qquad (\text{for } a>0)$	scaling
$\mathcal{L}\{f * g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$ (for $f(t), g(t)$ causal)	convolution
$\mathcal{L}\{tf(t)\} = -F'(s)$ (for $f(t)$ causal)	multiplication by t
$\mathcal{L}\{t^{-1}f(t)\} = \int_s^\infty F(u)du$ (for $f(t)$ causal)	multiplication by t^{-1}
$\mathcal{L}\{\int_0^t f(u)du\} = \frac{1}{s}F(s)$	integration w.r.t. t

Table of standard Laplace transforms:

$f(t)$	$\mathcal{L}\{f(t)\}(s)$	Region of validity
$t^n \qquad (\text{for } n \geq 0)$	$\frac{n!}{s^{n+1}}$	$s > 0$
$\sin(kt)$	$\frac{k}{s^2+k^2}$	$s > 0$
$\cos(kt)$	$\frac{s}{s^2+k^2}$	$s > 0$
$\sinh(kt)$	$\frac{k}{s^2-k^2}$	$s > k$
$\cosh(kt)$	$\frac{s}{s^2-k^2}$	$s > k$
$H(t-T) \qquad (\text{for } T \geq 0)$	$\frac{e^{-sT}}{s}$	$s > 0$
$\delta(t-T) \qquad (\text{for } T \geq 0)$	e^{-sT}	$s \in \mathbb{R}$

Fourier transform:

The Fourier transform and inverse Fourier transforms are given by:

$$\mathcal{F}\{f(t)\}(\omega) = F(\omega) := \int_{-\infty}^{\infty} f(t)e^{-i\omega t}dt, \qquad f(t) = \mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{i\omega t}d\omega.$$

Properties of the Fourier transform: $\mathcal{F}\{f(t)\} = F(\omega)$ in the following table:

$\mathcal{F}\{af(t)+bg(t)\} = a\mathcal{F}\{f(t)\} + b\mathcal{F}\{g(t)\}$	linearity
$\mathcal{F}\{e^{i\theta t}f(t)\} = F(\omega-\theta)$	frequency shift
$\mathcal{F}\{f(t-T)\} = e^{-i\omega T}F(\omega)$	time shift
$\mathcal{F}\{f^{(n)}(t)\} = (i\omega)^n F(\omega)$	differentiation
$\mathcal{F}\{F(t)\} = 2\pi f(-\omega)$	symmetry
$\mathcal{F}\{f(at)\} = \frac{1}{\lvert a\rvert}F(\frac{\omega}{a})$	scaling
$\mathcal{F}\{f * g(t)\} = \mathcal{F}\{f(t)\}\mathcal{F}\{g(t)\}$	convolution

Table of standard Fourier transforms:

$f(t)$	$\mathcal{F}\{f(t)\}(\omega)$
$\text{rect}_T(t)$	$\text{sinc}(\frac{T\omega}{2})$
$e^{-a\lvert t\rvert}$ (for $a > 0$)	$\frac{2a}{a^2+\omega^2}$
e^{-at^2} (for $a > 0$)	$\sqrt{\frac{\pi}{a}}e^{-\frac{\omega^2}{4a}}$
1	$2\pi\delta(\omega)$

The Fourier cosine transform: The Fourier cosine transform of an even function $f(t)$ is given by:

$$\mathcal{F}\{f(t)\}(\omega) = F(\omega) := \int_0^{\infty} f(t)\cos(\omega t)\ dt$$

and has inverse Fourier cosine transform

$$f(t) = \frac{2}{\pi}\int_0^{\infty} F(\omega)\cos(\omega t)\ d\omega.$$

The Fourier sine transform: The Fourier sine transform of an odd function $f(t)$ is given by:

$$\mathcal{F}\{f(t)\}(\omega) = F(\omega) := \int_0^{\infty} f(t)\sin(\omega t)\ dt$$

and has inverse Fourier cosine transform

$$f(t) = \frac{2}{\pi}\int_0^{\infty} F(\omega)\sin(\omega t)\ d\omega.$$

Integration:

Cylindrical polar coordinates

$$(x, y, z) = (r\cos(\theta), r\sin(\theta), z)$$
$$(r, \theta, z) = \left(\sqrt{x^2 + y^2}, \arctan(\tfrac{y}{x}), z\right)$$
$$dV = r dr d\theta dz.$$

Spherical polar coordinates

$$(x, y, z) = (\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi))$$
$$(\rho, \theta, \phi) = \left(\sqrt{x^2 + y^2 + z^2}, \arctan(\tfrac{y}{x}), \arccos(\tfrac{z}{\rho})\right)$$
$$dV = \rho^2\sin(\phi) d\rho d\phi d\theta.$$

Partial differential equations:

The wave equation is given by

$$\frac{\partial^2 u}{\partial t^2} = c^2 \frac{\partial^2 u}{\partial x^2} \qquad \text{(where } c > 0 \text{ is a constant)}$$

and has a general solution

$$u(x,t) = f(x+ct) + g(x-ct)$$

where f and g are arbitrary functions of a single variable.

D'Alembert's solution of the wave equation subject to the initial conditions

$$u(x,0) = \Phi(x), \qquad \frac{\partial u}{\partial t}(x,0) = \Psi(x)$$

is given by

$$u(x,t) = \frac{1}{2}\left[\Phi(x-ct) + \Phi(x+ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} \Psi(z)\ dz.$$