The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2018–2019**

Probability Modelling **2 hours**

Candidates should attempt ***ALL*** *four questions.*
The maximum marks for the various parts of the questions are indicated.
The paper will be marked out of 100. (Q1–26; Q2–19; Q3–31; Q4–24)

1 Individuals arriving at a border crossing are either EU citizens with probability p; or non-EU citizens with probability $1-p$, independently of one another. Define a delayed renewal process by saying that a renewal happens every time a sequence of two EU citizens in a row arrive after the beginning of the day.

(a) Let v_n be the probability that a renewal occurs on arrival of the n^{th} individual. Explain why $v_1 = 0$ and $v_n = p^2$ for $n \geq 2$. Hence find the generating function $V(s)$, defined as $\sum_{n=0}^{\infty} v_n s^n$ for $|s| < 1$. ***(6 marks)***

(b) Let u_n be the probability that, given that a renewal occurred on arrival of the t^{th} individual, another renewal occurs on arrival of the $t + n^{\text{th}}$. Give the values of u_0, u_1, and the value of u_n for $n \geq 2$, giving reasons for your answers. Hence find the generating function $U(s)$, defined as $\sum_{n=0}^{\infty} u_n s^n$ for $|s| < 1$. ***(7 marks)***

(c) Using the result that, in a delayed renewal process, $V(s) = U(s)B(s)$, where $B(s)$ is the probability generating function of the time until the first renewal, find the expected number of arrivals until the first renewal occurs. ***(6 marks)***

(d) What is the expected number of arrivals, after the start of the day, until the m^{th} renewal? ***(7 marks)***

2 A warehouse stores up to 5 copies of a game on any day. Let S_i be the level of stock at the beginning of day i. At the end of each day, an order for D_i games is placed at the warehouse, where each D_i is independently and identically distributed according to

$$D_i = \begin{cases} 0 & \text{with probability } \dfrac{1}{2} \\ 1 & \text{with probability } \dfrac{1}{4} \\ 2 & \text{with probability } \dfrac{1}{4} \end{cases}$$

As far as is possible, the order is immediately completed from the current level of stock. If the order quantity is greater than or equal to the existing stock, then the stock is immediately topped back up to 5 for the beginning of the next day. We model (S_i) for $i \geq 0$ as a Markov chain with state space $\{1,2,3,4,5\}$.

(a) Give the transition matrix of the Markov chain. ***(5 marks)***

(b) Given that the warehouse starts with 4 games in stock on day 0, find the expected number of days until they have to restock (i.e. need to top the stock back up to 5). ***(7 marks)***

(c) Given that the warehouse starts with 3 games in stock on day 0, find the probability that stock levels at the beginning of a day will be 1 before they will be 4. ***(7 marks)***

3 (a) A discrete time Markov chain (X_n) has state space $S = \{1,2,3,4\}$ and transition matrix

$$P_X = \begin{pmatrix} 0 & \frac{1}{3} & 0 & \frac{2}{3} \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & \frac{1}{2} & 0 & 0 \end{pmatrix}.$$

(i) Find all stationary distributions of the Markov chain. ***(6 marks)***

(ii) Deduce that, for all $i \in S$, $P(X_n = i)$ converges to a limit as $n \to \infty$, and give the value of the limit. ***(9 marks)***

(b) Another discrete time Markov chain (Y_n) has state space $S = \{1,2,3,4,5,6\}$ and transition matrix

$$P_Y = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & \frac{1}{2} & 0 & \frac{1}{2} & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ \frac{1}{3} & 0 & 0 & 0 & 0 & \frac{2}{3} \\ 0 & 0 & 0 & 0 & \frac{3}{4} & \frac{1}{4} \end{pmatrix}.$$

(i) Find the communicating classes of the Markov chain and state whether they are open or closed. ***(6 marks)***

(ii) Find the period of each state. ***(4 marks)***

(iii) Assume that the chain starts in state 2 at time 0. Prove that the distribution of the chain after an even number of moves, i.e. $\boldsymbol{\pi}^{2k}$ for $k = 0, 1, \ldots$, satisfies

$$\boldsymbol{\pi}^{2k} = (0, p_k, 0, 1 - p_k, 0, 0),$$

where

$$p_k = \frac{1}{3}\left[2 + \left(-\frac{1}{2}\right)^k\right].$$

(6 marks)

4 A Sheffield call centre for an insurance company is open for 12 hours per day. Treating the call centre's opening hours as the interval $(0, 12]$ with time measured in hours, car insurance claims arrive at the centre according to a variable rate Poissson process with rate $\lambda(t) = \frac{1}{100}(72t - 6t^2)$ for $t \in (0, 12]$.

(a) What is the distribution of the number of car insurance claims made in the day? ***(4 marks)***

(b) What is the probability that exactly 4 car insurance claims will be made in the first two hours of opening? ***(6 marks)***

(c) Given that exactly 2 car insurance claims are made in the first two hours of opening, find the conditional probability that none of them arrive in the first hour. ***(6 marks)***

(d) The insurance company also has a smaller London call centre that deals with lorry insurance claims. This London centre has the same opening hours, and lorry insurance claims at this London centre arrive according to a constant rate Poissson process with rate $\lambda(t) = 1/5$ per hour while it is open, i.e. for $t \in (0, 12]$. Assume that the processes of car and lorry insurance claims are independent.

(i) Find the probability that, in the first two hours of opening the company receives exactly 4 car insurance claims and 1 lorry insurance claim. ***(4 marks)***

(ii) What is the distribution of the total number of insurance claims made to the company (both car and lorry) up until hour t, for $0 < t \leq 12$? ***(4 marks)***

End of Question Paper