The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2018-2019**

Mechanics and Fluids **2 hours**

Attempt all four questions. The allocation of marks is shown in brackets.

1 (i) Calculate $\boldsymbol{\nabla}\phi$ and $\nabla^2\phi$ for the case

$$\phi = (y + 3z)\, f(x)\,, \tag{1}$$

where $f(x)$ is an arbitrary function of x.

Hence find the general solution for ϕ of the form given in (1) that satisfies the Laplace equation, $\nabla^2\phi = 0$.

(8 marks)

(ii) Consider $\boldsymbol{u} = 2\,y\,\boldsymbol{i} + 3\,\mathrm{e}^{3x}\boldsymbol{j}$. Calculate $\boldsymbol{\nabla}\times\boldsymbol{u}$, $(\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}$, $\boldsymbol{\nabla}(\boldsymbol{u}\cdot\boldsymbol{u})$ and verify that the following identity holds:

$$\boldsymbol{u}\times(\boldsymbol{\nabla}\times\boldsymbol{u}) = \frac{1}{2}\boldsymbol{\nabla}(\boldsymbol{u}\cdot\boldsymbol{u}) - (\boldsymbol{u}\cdot\boldsymbol{\nabla})\boldsymbol{u}\,. \tag{2}$$

(8 marks)

(iii) *Using suffix notation* and the property

$$\epsilon_{ijk}\,\epsilon_{klm} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}\,,$$

show that the identity (2) in part (ii) holds for any $\boldsymbol{u}$. ***(9 marks)***

2 (i) A tennis racket has density per unit length $\sigma(x) = (2 - \mathrm{e}^x)\,\mathrm{kg/m}$ over its length from $x = 0$ to $x = 0.5\,\mathrm{m}$. Find the total mass of the racket M, and the position of its centre of mass $\bar{x}$.

(11 marks)

(ii) Earth's solid mantle has spherical radius r that extends over $3L \le r \le 6L$, and has density

$$\rho(r) = \left(8 - \frac{2}{3L}\,r\right)\rho_0\,,$$

where L is a constant length scale and ρ_0 is a constant density scale. Show that the moment of inertia of the mantle about the z-axis can be expressed in spherical polar coordinates (SPs) as

$$I \;=\; \rho_0 \int \left(8\,r^2 - \frac{2}{3L}\,r^3\right)(1 - \cos^2\theta)\,\mathrm{d}V\,.$$

I takes the form $I = \alpha\,\rho_0\,L^5$, where α is a numerical constant. Compute the integral to find the value of α to 2 significant figures.

Over a period of 1 week $\approx 6 \times 10^5\,\mathrm{s}$, it is observed that the rate of rotation about the z-axis increases by $2 \times 10^{-12}\,\mathrm{rad/s}$. Calculate the torque T required to cause the change, assuming that T is constant over this period of time.

(14 marks)

3 (i) A particle subject to a force

$$\boldsymbol{F} = x^2\,y\,\boldsymbol{i} + 4\,y\,\boldsymbol{j}\,,$$

travels along a path consisting of two straight lines, first from $A = (0,1)$ to $C = (3,1)$, then from C to $B = (3,2)$. Calculate W_1, the work done by the force on the particle along this path ACB.

State Stokes' Theorem. Calculate the surface integral of the theorem for the triangular surface enclosed by the path $ACBA$, where BA follows the line $y = 1 + x/3$.

Use the result to find W_2, the work done by the force along the direct straight line from A to B.

(14 marks)

(ii) Consider the force

$$\boldsymbol{F} = \frac{1}{r}(1 - \cos\theta)\,\hat{\boldsymbol{r}} + \frac{\ln r}{r}\sin\theta\,\hat{\boldsymbol{\theta}}\,.$$

in cylindrical polar coordinates (CPs). Calculate the work done by the force on a particle travelling along the semi-circular path in the plane $z = 0$, $y \ge 0$ from $(x, y) = (2, 0)$ to $(-2, 0)$ by calculating the line integral.

Calculate a potential for the force and verify that it gives a result consistent with the work done along the semi-circular path.

(11 marks)

4 A sphere of radius a, centred on the origin, is fixed within an incompressible flow. Given the velocity potential in spherical polar coordinates (SPs),

$$V = U\left(r + \frac{a^3}{2r^2}\right)\cos\theta,$$

find the velocity field $\mathbf{u} = \boldsymbol{\nabla} V$.

Explain why the flow must be irrotational, and explain why the potential satisfies $\nabla^2 V = 0$.

Verify that the appropriate boundary conditions are satisfied at $r = a$, and that the velocity far from the sphere is $U\mathbf{k}$.

(12 marks)

Use Bernoulli's integral to find an expression for the pressure p in terms of r, θ, a, ρ, U and p_0, where $p \to p_0$ as $r \to \infty$.

Explain why F_3, the z-component of the force exerted by the fluid on the sphere, is

$$-\int_S p\cos\theta\,\mathrm{d}S,$$

where S is the surface of the sphere. [You do not have to compute the integral.]

(11 marks)

When the integral is evaluated it is found that $F_3 = 0$. Comment briefly on this result.

(2 marks)

[*For interest only:* In Question 2(ii) on Earth's mantle, $L = 1000\,\mathrm{km}$ and $\rho_0 = 1000\,\mathrm{kg/m^3}$. Angular momentum exchanges with the atmosphere and liquid iron outer core lead to changes in the length of day of the order of a few milliseconds, adding up to several seconds over a year.]

End of Question Paper

VECTOR CALCULUS IDENTITIES

$$\nabla(\phi + \psi) = \nabla\phi + \nabla\psi \tag{E.1}$$

$$\nabla \cdot (\mathbf{A} + \mathbf{B}) = \nabla \cdot \mathbf{A} + \nabla \cdot \mathbf{B} \tag{E.2}$$

$$\nabla \times (\mathbf{A} + \mathbf{B}) = \nabla \times \mathbf{A} + \nabla \times \mathbf{B} \tag{E.3}$$

$$\nabla(\phi\,\psi) = \phi\,\nabla\psi + \psi\,\nabla\phi \tag{E.4}$$

$$\nabla \cdot (\phi\,\mathbf{v}) = \phi\,\nabla \cdot \mathbf{v} + (\mathbf{v} \cdot \nabla)\phi \tag{E.5}$$

$$\nabla \times (\phi\,\mathbf{v}) = \phi\,\nabla \times \mathbf{v} + \nabla\phi \times \mathbf{v} \tag{E.6}$$

$$\nabla \cdot (\mathbf{u} \times \mathbf{v}) = \mathbf{v} \cdot \nabla \times \mathbf{u} - \mathbf{u} \cdot \nabla \times \mathbf{v} \tag{E.7}$$

$$\nabla \times (\mathbf{u} \times \mathbf{v}) = (\mathbf{v} \cdot \nabla)\mathbf{u} - (\mathbf{u} \cdot \nabla)\mathbf{v} + \mathbf{u}(\nabla \cdot \mathbf{v}) - \mathbf{v}(\nabla \cdot \mathbf{u}) \tag{E.8}$$

$$\nabla(\mathbf{u} \cdot \mathbf{v}) = (\mathbf{v} \cdot \nabla)\mathbf{u} + (\mathbf{u} \cdot \nabla)\mathbf{v} + \mathbf{v} \times (\nabla \times \mathbf{u}) + \mathbf{u} \times (\nabla \times \mathbf{v}) \tag{E.9}$$

- (E.1)-(E.3) express the linearity property of the vector operators.
- (E.4)-(E.7) follow immediately using subscript notation and the product rule. You should know or be able to derive them quickly, e.g.

$$\nabla \cdot (\phi\mathbf{v}) = \partial_i(\phi v_i) = \phi\,\partial_i v_i + v_i\partial_i\phi = \phi\,\nabla \cdot \mathbf{v} + (\mathbf{v} \cdot \nabla)\,\phi\,.$$

- (E.8)-(E.9) you should be able to derive, given the identity

$$\epsilon_{ijk}\,\epsilon_{klm} = \delta_{il}\,\delta_{jm} - \delta_{im}\,\delta_{jl}\ \ .$$

OPERATORS IN CYLINDRICAL POLARS (CPs)
AND SPHERICAL POLARS (SPs)

1. CPs (Cylindrical Polars) $\quad (r, \theta, z) \quad h_1 = 1,\ h_2 = r,\ h_3 = 1$

$$\nabla V = \frac{\partial V}{\partial r}\ \hat{\boldsymbol{r}} + \frac{1}{r}\frac{\partial V}{\partial \theta}\ \hat{\boldsymbol{\theta}} + \frac{\partial V}{\partial z}\ \hat{\boldsymbol{z}} \tag{CP.1}$$

$$\nabla \cdot \mathbf{F} = \frac{1}{r}\ \frac{\partial}{\partial r}(rF_1) + \frac{1}{r}\frac{\partial F_2}{\partial \theta} + \frac{\partial F_3}{\partial z} \tag{CP.2}$$

$$\nabla \times \mathbf{F} = \left[\frac{1}{r}\frac{\partial F_3}{\partial \theta} - \frac{\partial F_2}{\partial z}\right] \hat{\boldsymbol{r}} + \left[\frac{\partial F_1}{\partial z} - \frac{\partial F_3}{\partial r}\right] \hat{\boldsymbol{\theta}} + \left[\frac{1}{r}\frac{\partial}{\partial r}(rF_2) - \frac{1}{r}\frac{\partial F_1}{\partial \theta}\right] \hat{\boldsymbol{z}} \tag{CP.3}$$

$$\nabla^2 V = \frac{1}{r}\ \frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\ \frac{\partial^2 V}{\partial \theta^2} + \frac{\partial^2 V}{\partial z^2} \tag{CP.4}$$

2. SPs (Spherical Polars) $\ (r, \theta, \phi) \quad h_1 = 1,\ h_2 = r,\ h_3 = r\sin\theta$

$$\nabla V = \frac{\partial V}{\partial r}\ \hat{\boldsymbol{r}} + \frac{1}{r}\ \frac{\partial V}{\partial \theta}\ \hat{\boldsymbol{\theta}} + \frac{1}{r\sin\theta}\ \frac{\partial V}{\partial \phi}\ \hat{\boldsymbol{\phi}} \tag{SP.1}$$

$$\nabla \cdot \mathbf{F} = \frac{1}{r^2}\ \frac{\partial}{\partial r}(r^2 F_1) + \frac{1}{r\sin\theta}\ \frac{\partial}{\partial \theta}\ \{(\sin\theta)F_2\} + \frac{1}{r\sin\theta}\ \frac{\partial F_3}{\partial \phi} \tag{SP.2}$$

$$\begin{aligned}\nabla \times \mathbf{F} = \frac{1}{r\sin\theta}\ \left[\frac{\partial}{\partial \theta}\{(\sin\theta)F_3\} - \frac{\partial F_2}{\partial \phi}\right] \hat{\boldsymbol{r}} + \frac{1}{r\sin\theta}\ \left[\frac{\partial F_1}{\partial \phi} - \frac{\partial}{\partial r}\{(r\sin\theta)F_3\}\right] \hat{\boldsymbol{\theta}} \\ + \frac{1}{r}\ \left[\frac{\partial}{\partial r}(rF_2) - \frac{\partial F_1}{\partial \theta}\right] \hat{\boldsymbol{\phi}}\end{aligned} \tag{SP.3}$$

$$\nabla^2 V = \frac{1}{r^2}\ \frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2\sin\theta}\ \frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} \tag{SP.4}$$