The University Of Sheffield.

**SCHOOL OF MATHEMATICS AND STATISTICS** **Spring Semester 2018–2019**

**Fluid Mechanics I** **2 hours**

*Answer all four questions.*

**Please leave this exam paper on your desk**
**Do not remove it from the hall**

Registration number from U-Card (9 digits)
to be completed by student

| | | | | | | | | |
|---|---|---|---|---|---|---|---|---|
| | | | | | | | | |

Blank

**1** (i) Consider an incompressible fluid of velocity $\boldsymbol{v}$ with constant density $\rho$. Write down the momentum equation for this flow and give a brief explanation of the terms in this equation (you can ignore the body force for this part). ***(4 marks)***

(ii) Assuming that the characteristic length and velocity scales of the incompressible flow are $L$ and $U$, respectively, derive the expression for the Reynolds number $Re$ and explain briefly its physical meaning. ***(3 marks)***

(iii) Write down the mass conservation law for an incompressible fluid of velocity $\boldsymbol{v}$ with constant density. ***(1 mark)***

(iv) Consider the following two flows (both in Cartesian coordinates):
(a)

$$\boldsymbol{v} = (-\Omega y, \Omega x, 0),$$

(b)

$$\boldsymbol{v} = \left(-\frac{\Omega y}{x^2+y^2}, \frac{\Omega x}{x^2+y^2}, 0\right) \qquad \text{for } x^2+y^2 \neq 0,$$

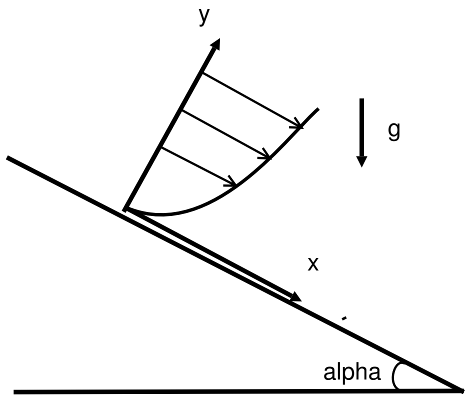
where $\Omega$ is constant. Compute the vorticity in each case and state which flow is rotational and which is irrotational. ***(7 marks)***

(v) Consider an incompressible fluid of velocity $\boldsymbol{v}$ with constant density $\rho$. Write down the expression for the total (material) derivative $\frac{D\boldsymbol{v}}{Dt}$ in terms of $\frac{\partial}{\partial t}$ and $\boldsymbol{v}$. Hence using tensor index notations and the Einstein summation convention, derive the following acceleration formula

$$\frac{D\boldsymbol{v}}{Dt} = \frac{\partial \boldsymbol{v}}{\partial t} + \nabla\left(\frac{|\boldsymbol{v}|^2}{2}\right) + (\nabla \times \boldsymbol{v}) \times \boldsymbol{v}.$$

***(10 marks)***

**2** Incompressible viscous fluid of uniform density $\rho$ flows steadily under gravity down a plane inclined at an angle of $\alpha = \frac{\pi}{3}$ to the horizontal. The fluid layer is of finite thickness $h$ and has a free surface on the top of the fluid. The free surface is assumed to remain parallel to the inclined plane. The fluid is Newtonian and has a kinematic viscosity $\nu$ (constant).




(i) Using the co-ordinate system shown in the figure, derive the $x$- and $y$-components of the momentum equations. Show that the pressure $p$ is given by

$$p = -\frac{\rho g y}{2} + f(x),$$

where $g$ is the gravitational acceleration and $f(x)$ is an arbitrary function of $x$. You may assume that $\frac{\partial u}{\partial x} = 0$, where $u$ is the component of velocity in the $x$-direction.

***(8 marks)***

(ii) State the boundary conditions on the free surface at $y = h$. ***(2 marks)***

(iii) Given that the atmospheric pressure $p_0$ is a constant show that

$$p - p_0 = \frac{\rho g (h - y)}{2}.$$

Show also that the profile of the component of the velocity in the $x$-direction is given by

$$u = \frac{\sqrt{3} g}{4\nu} y(2h - y).$$

***(8 marks)***

(iv) Calculate the volume flux down the plane (per unit length across any fixed plane perpendicular to the motion). ***(3 marks)***

(v) Calculate the drag/unit area exerted by the fluid on the bottom and top of the fluid layer. ***(4 marks)***

**3** (i) In cylindrical polar coordinates $(r, \theta, z)$, the fluid velocity $\boldsymbol{v}$ in the *steady* flow of an incompressible fluid has components

$$\left(-\frac{1}{2}\alpha r, v(r), \alpha z\right)$$

where $\alpha$ is a positive constant. For $\boldsymbol{v} = (v_r, v_\theta, v_z)$, you can use the following identities

$$(\boldsymbol{v} \cdot \nabla) = v_r \frac{\partial}{\partial r} + \frac{v_\theta}{r}\frac{\partial}{\partial \theta} + v_z \frac{\partial}{\partial z},$$

$$\nabla^2 = \frac{1}{r}\frac{\partial}{\partial r}\left(r \frac{\partial}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{\partial^2}{\partial z^2},$$

and

$$\nabla \times \boldsymbol{v} = \frac{1}{r}\begin{vmatrix} \boldsymbol{e}_r & r\boldsymbol{e}_\theta & \boldsymbol{e}_z \\ \dfrac{\partial}{\partial r} & \dfrac{\partial}{\partial \theta} & \dfrac{\partial}{\partial z} \\ v_r & r v_\theta & v_z \end{vmatrix}.$$

(a) Show that the vorticity $\boldsymbol{\omega}$ satisfies

$$\boldsymbol{\omega} = (0, 0, \omega(r)), \text{ where } \omega(r) = \frac{1}{r}\frac{\partial}{\partial r}(rv).$$

***(4 marks)***

(b) Hence, show that

$$\frac{\nu}{r}\frac{d}{dr}\left(r\frac{d\omega}{dr}\right) + \frac{\alpha}{r}\frac{d}{dr}\left(\frac{1}{2}r^2\omega\right) = 0.$$

***(10 marks)***

(ii) A solid sphere of radius $a$ and center O is moving with constant velocity $\boldsymbol{V}$ in a viscous fluid. You are given that the velocity of this fluid around the sphere takes the following form

$$\boldsymbol{v} = \left(\frac{a}{2r} + \frac{a^3}{2r^3}\right)\boldsymbol{V} + \left(\frac{a}{4r^3} - \frac{a^3}{4r^5}\right)(\boldsymbol{V} \cdot \boldsymbol{x})\boldsymbol{x},$$

where $\boldsymbol{x}$ is the position vector, and $r$ is the radial distance ($r \geq a$) in spherical coordinates. Calculate $\nabla \cdot \boldsymbol{v}$ and determine if $\boldsymbol{v}$ is incompressible or not.

***(11 marks)***

**4** For an incompressible flow with constant density $\rho$, the velocity $\boldsymbol{v}$ and pressure $p$ have mean and fluctuating parts as follows:

$$\boldsymbol{v} = \boldsymbol{V} + \boldsymbol{u},$$

$$p = P + p',$$

where $\langle \boldsymbol{v} \rangle = \boldsymbol{V}$, $\langle \boldsymbol{u} \rangle = 0$, $\langle p \rangle = P$ and $\langle p' \rangle = 0$. Here, the angular brackets denote the ensemble average. You may assume that there is no body force

(i) From $\nabla \cdot \boldsymbol{v} = 0$, derive the divergence of $\boldsymbol{V}$ and $\boldsymbol{u}$. ***(4 marks)***

(ii) From the Navier Stokes equation for $v_i$, derive the evolution equation for $V_i$.

***(9 marks)***

(iii) Consider a scalar field $\phi$ which evolves as

$$\left[\frac{\partial}{\partial t} + \boldsymbol{v} \cdot \nabla\right] \phi = \eta \nabla^2 \phi.$$

Here $\eta$ is a non-negative constant. Derive the evolution equation for $\langle \phi \rangle$ (the mean component of $\phi$). ***(12 marks)***

## End of Question Paper