The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring semester 2018-2019**

Operations Research **2 Hours**

Attempt all FOUR questions.

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Do not remove it from the hall

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1 (i) Use the two-phase method to find the optimal solution for the following linear programming problem:

$$\max \quad z = -2x_1 - 4x_2$$

subject to $x_1, x_2 \geq 0$ and

$$x_1 + x_2 \leq 5,$$
$$3x_1 + 2x_2 \geq 6.$$

State clearly your final optimal solution.

Hint: not counting the preprocessing step, you need only one simplex iteration in phase 1. ***(15 marks)***

(ii) Use the dual simplex method to find the optimal solution for the problem stated in Part (i). ***(10 marks)***

2 Company M mixes four ingredients and a filler to produce a nutritional powder. The powder contains three nutrients: A, B and C. The nutrient contents (unit: grams per kilogram of the ingredient) and the costs (unit: pounds per kilogram of the ingredient) of each ingredient are given in this table:

	A (g/kg)	B (g/kg)	C (g/kg)	Cost (£/kg)
Ingredient 1	1	8	4	4
Ingredient 2	2	1	5	6
Ingredient 3	7	4	1	8
Ingredient 4	2	6	2	5

The following information is known:

- According to marketing regulations, if company M wants to claim that the powder contains a nutrient, then the amount of that nutrient must be no less than a minimum value. The minimum value is different for different nutrients. In one kilogram of the powder, the minimum value is 5 grams for nutrient A, 7 grams for B, and 4 grams for C.
- Company M intends to claim that the powder contains **at least two nutrients**. They have no preference for which ones.
- Company M plans to mix 5000 kilograms of the powder.
- A fixed set-up cost of £300 is incurred if either ingredient 2 or 4 are used.
- The chemical properties of the ingredients are such that, if both ingredients 1 and 3 are used, then ingredient 2 cannot be used.
- Company M purchases ingredient 1 and 2 from company P. Due to profitability considerations, company P does not accept small orders. Therefore, company M can purchase either no less than 100 kilograms of ingredient 1 alone, or no less than 120 kilograms of ingredient 2 alone, or no less than 290 kilograms of the two ingredients combined.
- Starch is used as the filler. Its nutritional contents and cost can be neglected. There is no separate constraint on how much it should be used.

Let x_1, x_2, x_3, and x_4 be the amounts (in kilograms) of the four ingredients in **1 kilogram** of the powder. Formulate the mixed integer linear programming problem from which one can find the optimal solution to minimise the total cost. **Note: do NOT try to find the numerical solution of the problem; you may need to introduce more variables**. *(25 marks)*

3 You are given the following maximisation problem:

$$\max \quad f(x_1, x_2) = -x_1^2 + 2x_1 + 4x_2 + 4, \quad \text{subject to} \quad 4x_2^2 \leq 1. \tag{1}$$

Consider this as the primal problem.

(i) Let y be the dual variable corresponding to the constraint in Equation (1). Define the Lagrangian dual function $v(y)$ and simplify it as far as you can.
(16 marks)

(ii) You are given that the maximum of $f(x_1, x_2)$ for the primal problem in Equation (1) is $f_{\max} = 7$. By finding the minimum of $v(y)$, show that the strong duality condition holds. ***(5 marks)***

(iii) Write down the KKT conditions for the problem defined in Equation (1).
(4 marks)

4 Fonex is a phone maker that makes two phones. The following information is known:

- The unit profit for phone 1 is £30. It requires one hour of work on assembly line 1 and two hours of work on assembly line 2.
- The unit profit for phone 2 is £20. It requires two hours of work on assembly line 1 and one hour of work on assembly line 2.
- Assembly line 1 can be used up to 40 hours per week. Assembly line 2 can be used up to 50 hours per week. These two conditions provide two constraints for the problem.

We define x_1 and x_2 as the numbers of phone 1 and phone 2 produced each week, respectively, and let x_3 and x_4 be the slack variables corresponding to the constraints on the availability of assembly lines 1 and 2. To maximise the total profit, one can formulate a linear programming problem as follows:

$$\max \quad z = 3x_1 + 2x_2 \qquad (\textbf{One unit of } z \textbf{ is £10})$$

subject to

$$x_1 + 2x_2 \leq 40$$
$$2x_1 + x_2 \leq 50$$
$$x_1, x_2 \geq 0$$

As a simplification, we allow x_1 and x_2 to be non-integer numbers. Solving the problem with the simplex method, we find the following optimal tableau:

	x_1	x_2	x_3	x_4	Solution
z	0	0	1/3	4/3	80
x_1	1	0	$-1/3$	2/3	20
x_2	0	1	2/3	$-1/3$	10

 Question 4 continued on next page

4 (continued)

(i) Using the information given in the optimal tableau, find the optimal solutions for the decision variables, the cost, and the dual variables. ***(3 marks)***

(ii) Using the data given in the optimal tableau, verify that the complementary slackness conditions are satisfied for both the primal problem in Part (i) and the corresponding dual problem. ***(4 marks)***

(iii) Determine the optimality range for the unit profit for phone 1. ***(8 marks)***

(iv) If assembly line 2 can be used up to 55 hours per week, what would be the new optimal solution for x_1 and x_2? ***(5 marks)***

(v) Fonex is considering making a third phone. The phone needs 2 hours of work on assembly line 1 and 2 hours of work on assembly line 2. Let the unit profit for the phone be £P. It is profitable for Fonex to make the phone only when P is bigger than a minimum value. Find this value. ***(5 marks)***

End of Question Paper