The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2018–2019**

Quantum Theory **2 Hours**

Answer all four questions.

Please leave this exam paper on your desk
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1 (i) A person with mass is 80 kg is moving with speed 6 km/hr. In SI units, calculate the momentum, de Broglie wavelength and frequency of this person. You are given that the Planck constant is $h = 6.626 \times 10^{-34}$ J s.

(6 marks)

(ii) Let $\hat{A}$ and $\hat{B}$ be self-adjoint operators. Determine whether the operators $(\hat{A}\hat{B} + \hat{B}\hat{A})$ and $(\hat{A}\hat{B} - \hat{B}\hat{A})$ are self-adjoint.

(6 marks)

(iii) A yellow street-lamp emits lights with a frequency 5×10^{14} Hz and has the power 100 W (W = Watt = J s^{-1}). What is the number of quanta emitted per second?

(4 marks)

(iv) Show that for the position $\hat{x}$ and momentum $\hat{p}$ operators, $[\hat{x}, \hat{p}] = i\hbar$.

(6 marks)

(v) For the position operator $\hat{x}$, calculate $[\cosh \hat{x}, \hat{x}]$.

(3 marks)

2 (i) Let $\hat{A}$ be an operator which does not depend explicitly on time t. Show that

$$i\hbar \frac{d}{dt} E_\psi(\hat{A}) = \left\langle \psi \middle| [\hat{A}, \hat{H}]\psi \right\rangle$$

where $\hat{H}$ is the Hamiltonian operator, and $E_\psi(\hat{A})$ denotes the expectation value of the operator $\hat{A}$ in a normalised state ψ.

(12 marks)

(ii) Consider a quantum free particle with the Hamiltonian $\hat{H} = \hat{p}^2/2m$. Using the results in Question 2(i), calculate $E_\psi(\hat{p})$ and $E_\psi(\hat{x})$ for a normalised state ψ. Here $\hat{p}$ and $\hat{x}$ are the momentum and position operators, respectively.

(13 marks)

3 (i) A free particle of mass m is confined to the one-dimensional region $-L \leq x \leq L$. You are given that the orthonormal eigenstate of the Hamiltonian $\hat{H}$ is

$$\phi_n = \sqrt{\frac{1}{L}} \sin\left(\frac{n\pi(x+L)}{2L}\right),$$

where $n = 1, 2, 3, \ldots$.

(a) Calculate the energy eigenvalue corresponding to ϕ_n.

(3 marks)

(b) At $t = 0$, the particle is in the state

$$\psi(x, t = 0) = \frac{-4i\phi_1 + 3\phi_2}{5}.$$

For $t > 0$, calculate:
the wavefunction $\psi(x, t)$;
the probabilities that the particle has energy E_1 and E_3;
the expectation value of $\hat{p}^2$, where $\hat{p}$ is the momentum operator.

(8 marks)

(ii) Consider the one-dimensional potential

$$V(x) = \begin{cases} 0 & x < 0, \\ V_0 & x > 0, \end{cases}$$

where $V_0 > 0$ is a constant. Suppose that at time $t = 0$, a beam of quantum particles of mass m with energy E $(> V_0)$ is coming from the right of the potential step.

(a) Show that the time-independent Schrödinger equation is $\frac{d^2\phi}{dx^2} + k^2\phi = 0$ for $x < 0$, while $\frac{d^2\phi}{dx^2} + l^2\phi = 0$ for $x > 0$. Here $k^2 = \frac{2mE}{\hbar^2} > 0$ and $l^2 = \frac{2m(E - V_0)}{\hbar^2} > 0$.

(2 marks)

(b) Solve the time-independent Schrödinger equation from Question 3(ii)(a) above.

(8 marks)

(c) Hence calculate the reflected and transmitted currents.

(4 marks)

4 The simple harmonic oscillator describes the motion of a particle of mass m attached to a spring with a spring constant κ. The Hamiltonian operator $\hat{H}$ for the simple harmonic oscillator is given by

$$\hat{H} = \frac{\hat{p}^2}{2m} + \frac{1}{2}m\omega^2\hat{x}^2.$$

Here, $\hat{p}$ and $\hat{x}$ are the momentum and position operator, respectively; $\omega = \sqrt{\frac{\kappa}{m}}$.

(i) State the physical meaning of $\frac{\hat{p}^2}{2m}$ and $\frac{1}{2}m\omega^2\hat{x}^2$.

(1 mark)

(ii) Let $m = \omega = \hbar = 1$ for the rest of the question and define an operator $\hat{A}$ as

$$\hat{A} = \sqrt{\frac{1}{2}}\left(\hat{x} + i\hat{p}\right).$$

Calculate the adjoint operator $\hat{A}^*$ of $\hat{A}$.

(2 marks)

(iii) Show that $[\hat{A}, \hat{A}^*] = 1$.

(3 marks)

(iv) By using that

$$\hat{H} = \hat{N} + \frac{1}{2},$$

where $\hat{N} = \hat{A}^*\hat{A}$ is the number operator, show that $[\hat{H}, \hat{A}^*] = \hat{A}^*$ and $[\hat{N}, \hat{A}] = -\hat{A}$.

(6 marks)

(v) You are given that $|n\rangle$ is an eigenfunction of $\hat{N}$, and that the eigenvalue of $\hat{N}$ corresponding to $|n\rangle$ is $n = 0, 1, 2, 3, \ldots$. Calculate the energy eigenvalue of $|n\rangle$.

(3 marks)

(vi) Hence, calculate $\hat{N}\hat{A}\,|n\rangle$ and interpret your results. Infer the physical meaning of $\hat{A}$.

(5 marks)

(vii) In the eigenstate $|n\rangle$, calculate the expectation value of the momentum. (For this part, you can use the fact that $\hat{A}^*|n\rangle = \sqrt{n+1}|n+1\rangle$.)

(5 marks)

End of Question Paper