The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Spring Semester 2018–2019

Mathematical Methods — 2 hours

Marks will be awarded for your best FOUR *answers. The marks awarded to each question or section of question are shown in italics. The total number of marks for the paper is 100.*

1 (a) Define $\Delta(x;\epsilon)$ for $\epsilon > 0$ by

$$\Delta(x;\epsilon) = \frac{\epsilon}{\pi(x^2+\epsilon^2)}.$$

Show that

$$\int_{-\infty}^{\infty} \Delta(x;\epsilon)\,dx = 1.$$ ***(4 marks)***

By considering $\lim_{\epsilon\to 0} \int_{-\infty}^{\infty} f(x)\,\Delta(x-a;\epsilon)\,dx$ for an arbitrary well-behaved function $f(x)$ and a constant a, deduce that we could define the delta function $\delta(x)$ by

$$\delta(x) = \lim_{\epsilon\to 0} \frac{\epsilon}{\pi(x^2+\epsilon^2)}.$$ ***(5 marks)***

(b) Define an integral $I(x;\epsilon)$ for real x and $\epsilon > 0$ by

$$I(x;\epsilon) = \int_{-\infty}^{\infty} e^{ikx-\epsilon|k|}\,dk.$$

Evaluate $I(x;\epsilon)$ and, using the result of part (a), deduce that

$$\int_{-\infty}^{\infty} e^{ikx}\,dk = 2\pi\delta(x).$$ ***(6 marks)***

Hence show that

$$f(x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{-ikx}\,\hat{f}(k)\,dk,$$

where $\hat{f}(k)$ is the Fourier transform of $f(x)$. ***(4 marks)***

(c) If $\hat{f}(k) = e^{-k^2/2}$, use the result of part (b) to find the corresponding $f(x)$ for real x. ***(6 marks)***

[You may use

$$\int_{-\infty}^{\infty} e^{-\frac{1}{2}(k+ix)^2}\,dk = \int_{-\infty}^{\infty} e^{-\frac{1}{2}k^2}\,dk \qquad \text{for real } x,$$

and

$$\int_{-\infty}^{\infty} e^{-v^2}\,dv = \sqrt{\pi}.$$]

2 A semi-infinite rod occupies the positive x-axis. The temperature, $T(x,t)$, in the rod satisfies

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} \qquad \text{for} \quad 0 < x < \infty, \quad 0 < t < \infty, \tag{1}$$

where $\kappa\,(> 0)$ is the constant conductivity. Initially the rod has zero temperature everywhere, i.e. $T(x,0) = 0$, and for $t > 0$ the end of the rod is maintained at a constant temperature T_0, i.e. $T(0,t) = T_0$. For $t > 0$, $T(x,t) \to 0$ as $x \to \infty$.

The Laplace transform of $T(x,t)$ is defined by

$$\tilde{T}(x,s) = \mathcal{L}\{T(x,t)\} = \int_0^\infty e^{-st}\, T(x,t)\, dt,$$

and exists for $\operatorname{Re} s > c$ for some constant $c > 0$.

(a) Show that

$$\mathcal{L}\left\{\frac{\partial T}{\partial t}\right\} = s\tilde{T}(x,s) - T(x,0)$$

for $\operatorname{Re} s > c$. ***(4 marks)***

(b) By taking the Laplace transform of (1), show that

$$\tilde{T}(x,s) = A(s)e^{\left(\frac{s}{\kappa}\right)^{1/2} x} + B(s)e^{-\left(\frac{s}{\kappa}\right)^{1/2} x}$$

for some functions A and B. ***(5 marks)***

By applying the conditions at $x = 0$ and as $x \to \infty$ for $\operatorname{Re}(s^{1/2}) > 0$, find $A(s)$ and $B(s)$ and, hence, $\tilde{T}(x,s)$. ***(6 marks)***

(c) Given that the Laplace transform of

$$\int_{\frac{a}{\sqrt{2t}}}^\infty e^{-\frac{1}{2}u^2}\, du \qquad \text{is} \qquad \sqrt{\frac{\pi}{2}}\,\frac{e^{-a\sqrt{s}}}{s},$$

where a is a positive constant, show that

$$T(x,t) = \sqrt{\frac{2}{\pi}}\, T_0 \int_{\frac{x}{\sqrt{2\kappa t}}}^\infty e^{-\frac{1}{2}u^2}\, du.$$ ***(2 marks)***

Check that the conditions as $t \to 0+$ for $x > 0$, at $x = 0$ for $t > 0$, and as $x \to \infty$ for $t > 0$ are satisfied. ***(6 marks)***

For fixed $x > 0$, find

$$\lim_{t \to \infty} T(x,t).$$ ***(2 marks)***

[You may use

$$\int_{-\infty}^\infty e^{-v^2}\, dv = \sqrt{\pi}.$$]

3 The function $y(x)$ satisfies the ordinary differential equation

$$x^2 y'' - 2y = \ln(1+x)$$

in $0 < x < 1$, with the boundary conditions $y = 0$ at $x = 0$ and at $x = 1$.

(a) By trying $y = x^n$, find the independent solutions of

$$x^2 y'' - 2y = 0.$$

(3 marks)

(b) Given that Green's function, $G(x;\xi)$, for the boundary-value problem given at the beginning of the question is continuous at $x = \xi$, and that $\partial G/\partial x$ has a discontinuity of size $1/\xi^2$ at $x = \xi$, show that

$$G(x;\xi) = \begin{cases} \dfrac{1}{3}\left(1 - \xi^{-3}\right) x^2 & 0 \leqslant x < \xi \\ \\ \dfrac{1}{3}\left(x^2 - x^{-1}\right) & \xi < x \leqslant 1. \end{cases}$$

(13 marks)

(c) Use Green's function to show that the solution to the boundary-value problem given at the beginning of the question is

$$y(x) = \frac{1}{3}\left(x^2 - x^{-1}\right)\{(1+x)\ln(1+x) - x\} + \frac{1}{3}x^2 \int_x^1 \left(1 - \xi^{-3}\right) \ln(1+\xi)\, d\xi.$$

(6 marks)

Use this to find $y'(x)$, and hence find $y'(1)$. ***(3 marks)***

4 Consider the equation

$$\epsilon x^3 + x^2 - 1 = 0, \tag{2}$$

where ϵ is a constant satisfying $0 < \epsilon \ll 1$.

(a) The solution to equation (2) can be written as

$$x = \frac{1}{\epsilon}\left(x_0 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \cdots\right),$$

where $x_0, x_1, x_2, \ldots$ are $O(1)$ as $\epsilon \to 0$.

Use this expression to show that one of the three solutions to equation (2) is

$$x = 1 - \frac{1}{2}\epsilon + O(\epsilon^2),$$

and to find the other two solutions, correct to order ϵ as $\epsilon \to 0$.

(19 marks)

(b) Given the rearrangement

$$x = -\frac{1}{\epsilon}\left(1 - \frac{1}{x^2}\right)$$

of equation (2), use iteration to find the solution close to $-1/\epsilon$, correct to order ϵ^3 as $\epsilon \to 0$. ***(6 marks)***

5 The Laplace transform, $\tilde{f}(s)$, of a function $f(t)$ is defined by

$$\tilde{f}(s) = \mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt,$$

where s is real and positive.

(a) By integrating by parts, show that if f is n times differentiable then

$$\tilde{f}(s) = \frac{f(0)}{s} + \frac{f'(0)}{s^2} + \cdots + \frac{f^{(n-1)}(0)}{s^n} + R_n(s), \tag{3}$$

where

$$R_n(s) = \frac{1}{s^n} \int_0^\infty e^{-st} f^{(n)}(t)\, dt.$$ ***(8 marks)***

$\left[\text{ You may assume that } \lim_{t\to\infty} \left\{e^{-st} f^{(m)}(t)\right\} = 0 \text{ for } m = 0, 1, \ldots, n-1. \right]$

(b) Use equation (3) to deduce that for $f(t) = \sin t$

$$\tilde{f}(s) = \frac{1}{s^2} - \frac{1}{s^4} + \frac{1}{s^6} - \frac{1}{s^8} \cdots + R_n(s).$$ ***(6 marks)***

Show that in this case

$$|R_n(s)| \leqslant \frac{1}{s^{n+1}}$$

and, hence, that the series is an asymptotic expansion as $s \to \infty$.

(6 marks)

Show that the exact result for $\tilde{f}(s)$ gives the same expansion as $s \to \infty$.

(5 marks)

End of Question Paper