Data provided: Formulae sheet. Graph paper available.

MAS340

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS

Spring Semester 2018-2019

Mathematics (Computational Methods)

Two hours

Attempt all FOUR questions

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student

Blank

1 (i) Classify the following differential equations as either elliptic, parabolic or hyperbolic:

(a) $$5\frac{\partial^2 u}{\partial x^2} + 6\frac{\partial^2 u}{\partial x \partial y} + 3\frac{\partial^2 u}{\partial y^2} = -4\frac{\partial u}{\partial x}$$ (1 mark)

(b) $$-\frac{\partial^2 u}{\partial x^2} + 3\frac{\partial^2 u}{\partial x \partial y} + 3\frac{\partial^2 u}{\partial y^2} = 12$$ (1 mark)

(c) $$\frac{\partial^2 u}{\partial x^2} + 4\frac{\partial^2 u}{\partial x \partial y} + 4\frac{\partial^2 u}{\partial y^2} = 6\frac{\partial u}{\partial y}$$ (1 mark)

(ii) Use Taylor series expansions to derive the following approximations:

$$\left(\frac{dy}{dx}\right)_{x=x_0} \approx \frac{y_1 - y_0}{h}, \qquad \left(\frac{d^2y}{dx^2}\right)_{x=x_0} \approx \frac{y_1 - 2y_0 + y_{-1}}{h^2},$$

where $y = y(x)$ is a continuous function and $y_i = y(x_0 + ih)$, $i = 0, \pm 1$. In each case state the order of the error term. (6 marks)

(iii) (a) Using the forward time and centred space (FTCS) approximation derive equations for an explicit approximation solution of the heat equation

$$\frac{\partial u}{\partial t} = 7\frac{\partial^2 u}{\partial x^2},$$

where $u = u(x, t)$, in the interval $0 \le x \le 1$ and $t \ge 0$, subject to the additional conditions $u(x, 0) = 5x^2 + 9x$, $u(0, t) = 0$, $\frac{\partial u}{\partial x}(1, t) = 0$.
The interval should be divided into 5 sections and a time-step of 0.004 should be used.

(6 marks)

(b) Set up a table showing the values of $u(x, t)$ at the grid points for $t = 0$ and extend the table to include the values where $t \le 0.008$.
(10 marks)

2 (i) (a) Describe briefly how the Crank-Nicolson scheme is derived.
(1 mark)

(b) The function $u(x, t)$ satisfies the heat conduction equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}$$

together with the conditions

$$u(x, 0) = 2x(1 - x), \quad 0 \leq x \leq 1,$$

$$u(0, t) = 0,$$

$$u(1, t) = 0.$$

You are given that the Crank-Nicolson scheme for this heat conduction equation has the form

$$-r u_{i-1,j+1} + (2+2r) u_{i,j+1} - r u_{i+1,j+1} = r u_{i-1,j} + (2-2r) u_{i,j} + r u_{i+1,j},$$

where, in the usual notation, $r = \frac{k}{h^2}$.
For the case where $h = 0.2$ and $k = 0.04$, formulate the Crank-Nicolson finite-difference equations for $i = 1, 2, 3, 4$ at the time-step $j = 0$.
Do not attempt to solve these equations. (9 marks)

 Question 2 continued on next page

2 (continued)

(ii) (a) Let $a = x_0 < x_1 < \cdots < x_N = b$, and let $f(x_i) = f_i$ for function f which is continuous on $[a, b]$. Describe the properties of the cubic spline interpolant S to f at the equally spaced points $x_i,\ i = 0, 1, \cdots, N$. (3 marks)

(b) The cubic interpolant defined in the interval $[x_i, x_{i+1}]$ has the form

$$s_i(x) = a_i(x - x_i)^3 + b_i(x - x_i)^2 + c_i(x - x_i) + d_i.$$

The coefficients of the interpolant polynomial are defined as:

$$a_i = \frac{\sigma_{i+1} - \sigma_i}{6h},$$

$$b_i = \frac{\sigma_i}{2},$$

$$c_i = \frac{f_{i+1} - f_i}{h} - \frac{h}{6}(2\sigma_i + \sigma_{i+1}),$$

$$d_i = f_i,$$

where $h = x_{i+1} - x_i$, f_i is the value of the function at the i-th position, $s''(x_i) = \sigma_i$, where $'$ denotes differentiation with respect to x, and $0 \leq i \leq N$.

Given that

$$\sigma_{i-1} + 4\sigma_i + \sigma_{i+1} = \frac{6}{h^2}\left(f_{i-1} - 2f_i + f_{i+1}\right),$$

determine the natural cubic spline approximation between the following data points

x	0	$\pi/8$	$\pi/4$	$3\pi/8$
$f(x)$	0	0.8058	1.7374	2.9404

Work correct to FOUR decimal places. (9 marks)

(c) Use the cubic spline interpolant to estimate $f(0.7)$ and $f'(0.7)$. (3 marks)

3 Use the branch and bound method to solve the following integer programming problem:

Maximise $z = 80x_1 + 45x_2$ subject to the constraints: $x_1 \geq 0$, $x_2 \geq 0$, $x_1 + x_2 \leq 7$, $12x_1 + 5x_2 \leq 60$, where x_1 and x_2 are integers.

Your solution should include:

- An appropriate graph to represent the possible set of solutions and the slope of the objective function. (3 marks)
- The solution to the linear programming problem associated with this integer programming problem. (3 marks)
- The full solution to the integer programming problem. (15 marks)
- A tree diagram to indicate the structure of your solution. (4 marks)

Work in fractions or with 3 decimal places.

4 (i) (a) Find and classify the stationary point of the function

$$f(x, y) = x^2 + y^2 + 8x + 8.$$

(6 marks)

(b) Calculate an approximation to the minimum of $f(x, y)$ using one iteration of Newton's method starting from the point $(2, 2)$. (7 marks)

(c) Describe briefly the advantages and disadvantages of the method of steepest descent and Newton's method for finding an approximation to a local minimum of a function of two variables. (2 marks)

(ii) Express the following constraints in a form suitable for the integer programming method:

'Either $x_1 + 2x_2 \leq 15$ or $6x_1 - 5x_2 \leq 1$'. (3 marks)

(iii) You are given that

$$5x_1 - 4x_2 + 3x_3 \leq 5, \quad 11x_1 + 2x_2 - x_3 \leq 6, \quad x_1 + 8x_2 - 9x_3 \leq 10.$$

Formulate the following sets of constraints in a form suitable for integer programming:

(a) All three constraints hold.

(b) Only one of the constraints holds.

(c) At least two of the constraints hold. (7 marks)

End of Question Paper

Formulae Sheet

Notation:

$$U(x_i,\ t_j) \equiv U_{i\,j}$$

Forward difference formula for $\partial U/\partial t$:

$$\frac{\partial U}{\partial t} \approx \frac{U_{i,j+1} - U_{ij}}{\Delta t}$$

Backward difference formula for $\partial U/\partial t$:

$$\frac{\partial U}{\partial t} \approx \frac{U_{ij} - U_{i,j-1}}{\Delta t}$$

Central difference formula for $\partial U/\partial x$:

$$\frac{\partial U}{\partial x} \approx \frac{U_{i+1,j} - U_{i-1,j}}{2\Delta x}$$

Central difference formula for $\partial^2 U/\partial x^2$:

$$\frac{\partial^2 U}{\partial x^2} \approx \frac{U_{i+1,j} - 2U_{ij} + U_{i-1,j}}{\Delta x^2}$$