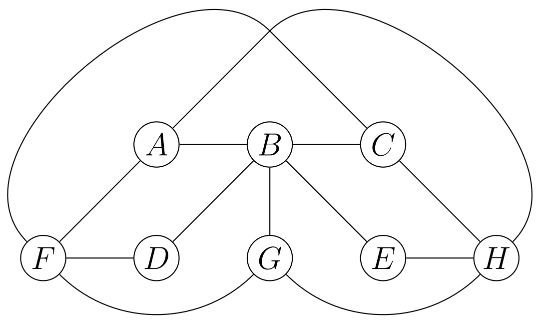
SCHOOL OF MATHEMATICS AND STATISTICS | Spring Semester 2018–2019

Graph Theory | **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

1 For this question, Γ refers to the graph shown below.



(i) Define what it means for a graph to be Hamiltonian, and prove that Γ is not Hamiltonian. **(4 marks)**

(ii) Define what it means for a graph to be semi-Eulerian, and prove that Γ is not semi-Eulerian. Find an edge $e \in \Gamma$ so that when we delete e the result $\Gamma \setminus e$ *is* semi-Eulerian. **(5 marks)**

(iii) Call a graph G *semi-demi-Eulerian* if there are two walks P and Q in Γ,

$$P = v_1 - v_2 - \cdots - v_k \quad \text{and} \quad Q = w_1 - w_2 - \cdots - w_\ell$$

such that P repeats no edges, Q repeats no edges, P and Q share no edges, and every edge of Γ is used in P or Q. Show that Γ is semi-demi-Eulerian. Prove furthermore that a connected graph G is semi-demi-Eulerian if and only if G has at most four vertices of odd degree. We allow P and Q to be closed or empty. (Hint: adapt either of the two proofs we saw for the semi-Eulerian case). **(9 marks)**

(iv) State the Handshaking Lemma, and use it to prove that if X is an unknown atom and $C_2X_2H_6$ is a tree, then X has valency 2. Finally, given that $C_2X_2H_6$ is a tree, draw all of its isomers. **(7 marks)**

2 For this question, Γ refers to a weighted complete graph on seven vertices $A, B, \ldots, G$. The edge weights of Γ are shown in the following table.

A						
7	B					
7	4	C				
8	3	5	D			
4	6	9	8	E		
9	8	8	7	7	F	
7	9	9	9	8	8	G

(i) List the edges of a cheapest spanning tree of H in the order they are added if the tree is built using Prim's algorithm beginning at vertex G.

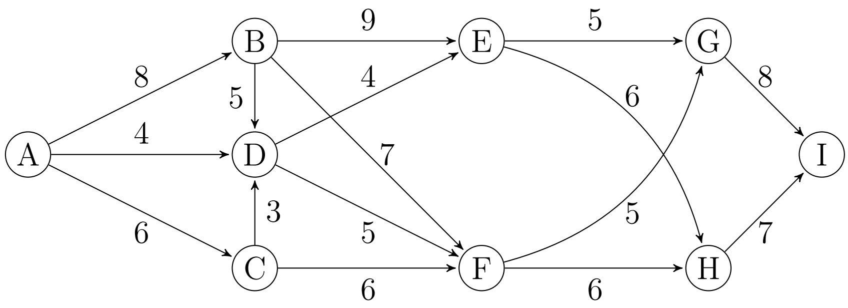
(4 marks)

(ii) State the Traveling Salesperson Problem (TSP). Describe the method used in lecture to obtain a lower bound for the TSP, and prove it actually produces a lower bound. Find an upper bound for the TSP for Γ by using the nearest neighbour heuristic starting at G, and find a lower bound for the TSP for Γ using the method from lecture, deleting F.

(7 marks)

(iii) Draw the tree with Prüfer code $2, 7, 1, 8, 2, 8, 1, 8$. **(4 marks)**

The last two parts refer to the weighted directed graph Z below:



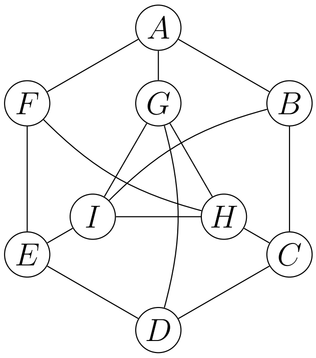
(iv) Give the lengths of all the *shortest* paths from A to any other vertex in Z, in the order the shortest paths are discovered by Dijkstra's algorithm.

(4 marks)

(v) Find the length L of the *longest* directed path from A to I in the graph Z. For which edges e would increasing $w(e)$ by 0.1 increase L? For which e would decreasing $w(e)$ by 0.1 decrease L?

(6 marks)

3 For this question, Γ refers to the graph shown below.



(i) State Kuratowski's theorem, and use it to prove that Γ is not planar. **(5 marks)**

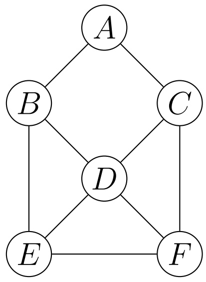
(ii) Using the planarity algorithm with the Hamiltonian cycle $ABCDEFHIGA$, give another proof that Γ is not planar. **(4 marks)**

(iii) Draw Γ on the torus so that no edges cross. Draw Γ on the Mobius band so that no edges cross. **(4 marks)**

(iv) Prove that if we delete any single edge from Γ, the result is still not planar. **(5 marks)**

(v) State Euler's theorem for graphs drawn on the sphere. Use it to prove the following theorem: If every vertex of a graph G has degree at least d, and G is drawn on the sphere so that every face has at least c sides, then $\frac{1}{c} + \frac{1}{d} > \frac{1}{2}$. **(7 marks)**

4 For this question, Γ refers to the graph shown below.



(i) Define the chromatic number $\chi(G)$ of a graph G, and compute the chromatic number of Γ, with justification. **(4 marks)**

(ii) Define the chromatic index $\chi'(G)$ of a graph G, and compute the chromatic index of Γ, with justification. **(4 marks)**

(iii) Define the chromatic polynomial $P_G(k)$ of a graph G, and compute the chromatic polynomial of Γ. **(5 marks)**

(iv) State the deletion-contraction formula, and use it to prove that the chromatic polynomial of any graph G is in fact a polynomial. **(5 marks)**

(v) A graph H has chromatic polynomial

$$P_H(k) = k(k-1)(k-2)^3(k-3)^2(k^2-4k+6)$$

What is the chromatic number of H? Prove that the chromatic index of H is at least 5. **(7 marks)**

End of Question Paper