1. For this question, \( \Gamma \) refers to the graph shown below.

(i) Define what it means for a graph to be Hamiltonian, and prove that \( \Gamma \) is not Hamiltonian. \(\text{(4 marks)}\)

(ii) Define what it means for a graph to be semi-Eulerian, and prove that \( \Gamma \) is not semi-Eulerian. Find an edge \( e \in \Gamma \) so that when we delete \( e \) the result \( \Gamma \setminus e \) is semi-Eulerian. \(\text{(5 marks)}\)

(iii) Call a graph \( G \) semi-demi-Eulerian if there are two walks \( P \) and \( Q \) in \( \Gamma \),

\[
P = v_1 - v_2 - \cdots - v_k \quad \text{and} \quad Q = w_1 - w_2 - \cdots - w_\ell
\]

such that \( P \) repeats no edges, \( Q \) repeats no edges, \( P \) and \( Q \) share no edges, and every edge of \( \Gamma \) is used in \( P \) or \( Q \). Show that \( \Gamma \) is semi-demi-Eulerian. Prove furthermore that a connected graph \( G \) is semi-demi-Eulerian if and only if \( G \) has at most four vertices of odd degree. We allow \( P \) and \( Q \) to be closed or empty. (Hint: adapt either of the two proofs we saw for the semi-Eulerian case). \(\text{(9 marks)}\)

(iv) State the Handshaking Lemma, and use it to prove that if \( X \) is an unknown atom and \( C_2X_2H_6 \) is a tree, then \( X \) has valency 2. Finally, given that \( C_2X_2H_6 \) is a tree, draw all of its isomers. \(\text{(7 marks)}\)
For this question, \( \Gamma \) refers to a weighted complete graph on seven vertices \( A, B, \ldots, G \). The edge weights of \( \Gamma \) are shown in the following table.

\[
\begin{array}{cccccc}
A & 7 & B \\
7 & 4 & C \\
8 & 3 & 5 & D \\
4 & 6 & 9 & 8 & E \\
9 & 8 & 8 & 7 & 7 & F \\
7 & 9 & 9 & 8 & 8 & G \\
\end{array}
\]

(i) List the edges of a cheapest spanning tree of \( H \) in the order they are added if the tree is built using Prim’s algorithm beginning at vertex \( G \).  

(4 marks)

(ii) State the Traveling Salesperson Problem (TSP). Describe the method used in lecture to obtain a lower bound for the TSP, and prove it actually produces a lower bound. Find an upper bound for the TSP for \( \Gamma \) by using the nearest neighbour heuristic starting at \( G \), and find a lower bound for the TSP for \( \Gamma \) using the method from lecture, deleting \( F \).  

(7 marks)

(iii) Draw the tree with Prüfer code 2, 7, 1, 8, 2, 8, 1, 8.  

(4 marks)

The last two parts refer to the weighted directed graph \( Z \) below:

(iv) Give the lengths of all the shortest paths from \( A \) to any other vertex in \( Z \), in the order the shortest paths are discovered by Dijkstra’s algorithm.  

(4 marks)

(v) Find the length \( L \) of the longest directed path from \( A \) to \( I \) in the graph \( Z \). For which edges \( e \) would increasing \( w(e) \) by 0.1 increase \( L \)? For which \( e \) would decreasing \( w(e) \) by 0.1 decrease \( L \)?  

(6 marks)
3 For this question, $\Gamma$ refers to the graph shown below.

(i) State Kuratowski’s theorem, and use it to prove that $\Gamma$ is not planar. (5 marks)

(ii) Using the planarity algorithm with the Hamiltonian cycle $ABCDEFGHIA$, give another proof that $\Gamma$ is not planar. (4 marks)

(iii) Draw $\Gamma$ on the torus so that no edges cross. Draw $\Gamma$ on the Mobius band so that no edges cross. (4 marks)

(iv) Prove that if we delete any single edge from $\Gamma$, the result is still not planar. (5 marks)

(v) State Euler’s theorem for graphs drawn on the sphere. Use it to prove the following theorem: If every vertex of a graph $G$ has degree at least $d$, and $G$ is drawn on the sphere so that every face has at least $c$ sides, then

$$\frac{1}{c} + \frac{1}{d} > \frac{1}{2}.$$ (7 marks)
For this question, $\Gamma$ refers to the graph shown below.

(i) Define the chromatic number $\chi(G)$ of a graph $G$, and compute the chromatic number of $\Gamma$, with justification. (4 marks)

(ii) Define the chromatic index $\chi'(G)$ of a graph $G$, and compute the chromatic index of $\Gamma$, with justification. (4 marks)

(iii) Define the chromatic polynomial $P_G(k)$ of a graph $G$, and compute the chromatic polynomial of $\Gamma$. (5 marks)

(iv) State the deletion-contraction formula, and use it to prove that the chromatic polynomial of any graph $G$ is in fact a polynomial. (5 marks)

(v) A graph $H$ has chromatic polynomial

$$P_H(k) = k(k - 1)(k - 2)^3(k - 3)^2(k^2 - 4k + 6)$$

What is the chromatic number of $H$? Prove that the chromatic index of $H$ is at least 5. (7 marks)

End of Question Paper