The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2018–2019**

Knots and Surfaces **2 hours and 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets. Strings, pipe cleaners, shoe laces or similar aids for making knots may be used.

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student

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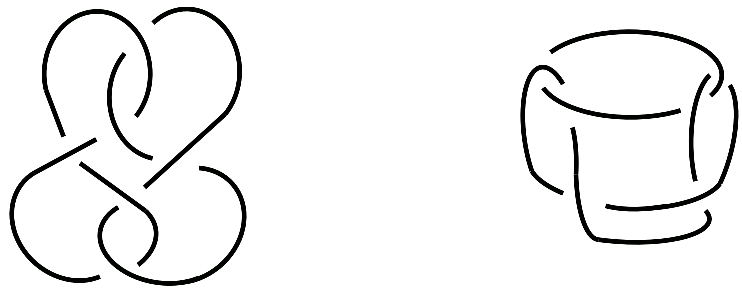
1 (i) (a) Define what it means for two link diagrams to be Reidemeister equivalent. Your answer should include the definition of the Reidemeister moves. **(6 marks)**

(b) State Reidemeister's Theorem. **(2 marks)**

(ii) Show with an explicit sequence of Reidemeister moves that the two link diagrams below are Reidemeister equivalent.



(7 marks)

(iii) Given an oriented link diagram D, define the *total linking number* $\mathrm{lk}(D)$ to be

$$\mathrm{lk}(D) := \frac{1}{2}\sum_{C} \epsilon(C),$$

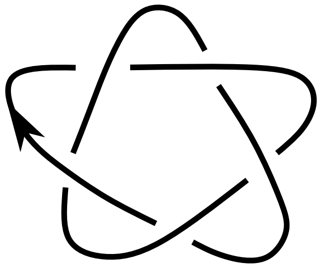
where the sum is taken over all crossings whose understrands and overstrands are from *different* components of D, and $\epsilon(C)$ denotes the sign of the crossing. Note that because we only sum over crossings which involve *different* components that, in general, $\mathrm{lk}(D) \neq \omega(D)$ where $\omega(D)$ is the writhe of a diagram, and that $\mathrm{lk}(D) = 0$ whenever D is a knot diagram.

(a) Prove that the linking number is invariant under each of the Reidemeister moves, and deduce that it is an invariant of oriented links. **(7 marks)**

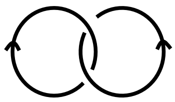
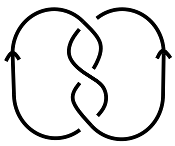
(b) Prove that there are infinitely many distinct oriented links with two components.
Hint: You might find it useful to look at the family of link diagrams in Question 2 Part (iii). **(3 marks)**

2 (i) State the Theorem which characterises the Jones polynomial in terms of the skein relation. Your answer should include a description of all oriented diagrams that occur in the relation. **(6 marks)**

(ii) Calculate the Jones polynomial of the cinquefoil knot shown below:

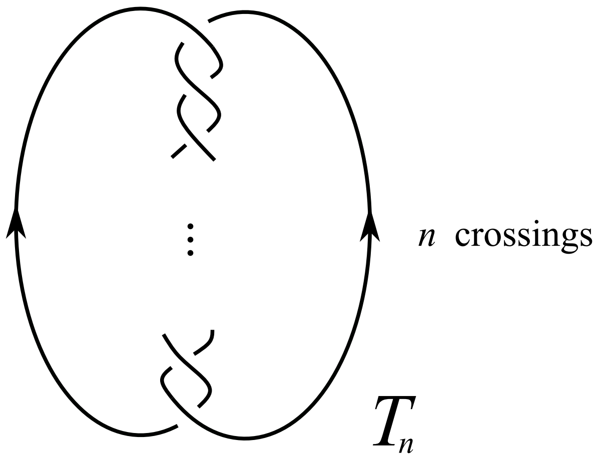


You may use the table of Jones polynomials below in your calculation.

D	$V_D(A)$
	$-A^{10} - A^2$
	$-A^{16} + A^{12} + A^4$

(9 marks)

(iii) For n a positive integer, let T_n denote the diagram below with n crossings.



(a) By considering the Kauffman bracket or otherwise, show that

$$V_{T_{n+1}}(A) \;=\; -A^{6n+4} - A^2 V_{T_n}(A).$$

(6 marks)

(b) Recall that a knot is said to be *chiral* if it is not equivalent to its mirror image. Show that there exist infinitely many distinct chiral knots. **(4 marks)**

3 (i) (a) Define the *cancellation move* on surface words.

(b) Define the *orientable handle move* on surface words.

(c) Define the *non-orientable handle move* on surface words.

(d) Give a diagrammatic justification of the cancellation move.

(8 marks)

(ii) (a) Draw plane models for the Klein bottle K and the torus T, and write down their surface words. **(4 marks)**

(b) Use surface word operations to express the surface $K\#T$ as a connected sum of projective planes. **(7 marks)**

(iii) Determine whether the following statements are true or false. Your answer should be fully justified with a rigorous proof or counterexample.

(a) $S^2\#\Sigma \cong \Sigma$ for every closed connected surface Σ.

(b) The function

$$f : [0, 2\pi) \to \{(\cos(\theta), \sin(\theta)) \mid \theta \in [0, 2\pi)\}$$

defined by

$$\theta \mapsto (\cos(\theta), \sin(\theta))$$

is a homeomorphism. **(6 marks)**

4 (i) (a) State the classification of closed surfaces. **(4 marks)**

(b) Express the connected sum of two Klein bottles as the connected sum of projective planes. Your answer should contain full justification.

(5 marks)

(ii) (a) State the inclusion-exclusion principle for the Euler characteristic.

(4 marks)

(b) Let K be a knot in $\mathbb{R}^3$ and Σ_1, Σ_2 be surfaces with $\Sigma_1 \cap \Sigma_2 = K$. Express $\chi(\Sigma_1 \cup \Sigma_2)$ in terms of $\chi(\Sigma_1)$ and $\chi(\Sigma_2)$. **(4 marks)**

(iii) (a) Show that if a torus has a covering pattern consisting only of triangles with the property that exactly n edges meet at each vertex then $n = 6$.

(5 marks)

(b) Exhibit a covering pattern of the torus with the property that 6 distinct edges meet at each vertex. **(3 marks)**

End of Question Paper