The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2018–2019**

MAS346 Groups and Symmetry **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

Additional Material: *Diagram for Question 2*

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Do not remove it from the hall

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to be completed by student

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1 (i) (a) Define an isometry of $\mathbf{R}^n$. Let Isom_n be the set of isometries of $\mathbf{R}^n$. Show that it is closed under composition and taking inverses.
(7 marks)

(b) Let $f \in \mathrm{Isom}_n$ be given by $f(x) = Ax + a$, where $A \in O_n$ and $a \in \mathbf{R}^n$. For $b \in \mathbf{R}^n$ describe the isometry fT_bf^{-1}, where T_b is the translation by b. ***(4 marks)***

(ii) Which of the following functions are isometries? [No justification required.]

(a) $f : \mathbf{R} \to \mathbf{R} : f(x) = x^2 + 3x + 5$.

(b) $f : \mathbf{R}^2 \to \mathbf{R}^2 : f(x, y) = (\sqrt{3}/2x - y/2 + 5, x/2 + \sqrt{3}/2y)$.

(c) $f : \mathbf{R}^2 \to \mathbf{R}^2 : f(x, y) = (1 + y, 1 + x)$.

(d) $f : \mathbf{R}^2 \to \mathbf{R}^2 : f(x, y) = (2x, 2y)$.

(4 marks)

(iii) Define the point homomorphism $\psi : \mathrm{Isom}_n \to O_n$ and show that it is a homomorphism. ***(3 marks)***

(iv) Consider now isometries of the plane (i.e. $n = 2$). For a line $L \subset \mathbf{R}^2$ and a vector $a \in \mathbf{R}^2$ parallel to L let $f = G_{L,a} \in \mathrm{Isom}_2$ be the corresponding glide-reflection.

(a) Show that $f^2 = T_{2a}$. ***(2 marks)***

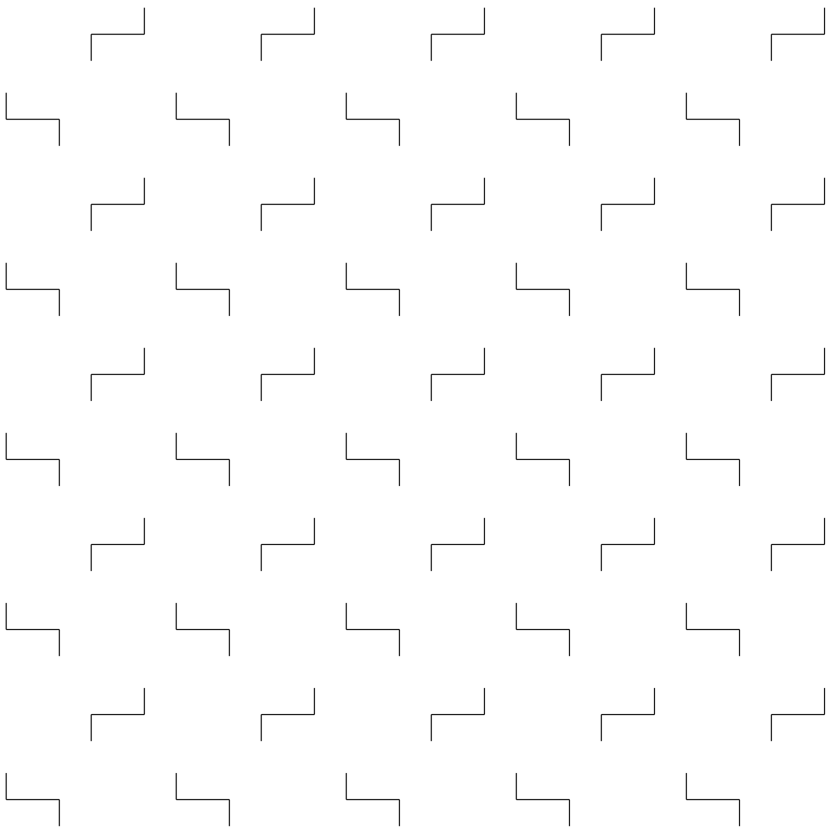
(b) Prove that $L = \{x \in \mathbf{R}^2 \mid f(x) = x + a\}$. ***(3 marks)***

(c) Show that $\psi(f) = S_\theta$, where $\theta/2$ is the angle between the x-axis and L. ***(2 marks)***

2 (i) For any subgroup $H \leq \text{Isom}_2$ recall that $\psi(H) \leq O_2$ is its point group and $\text{Trans}(H) \leq \mathbf{R}^2$ its translation subgroup. Explain which properties $\psi(H)$ and $\text{Trans}(H)$ need to satisfy so that the group H is a wallpaper group.

(3 marks)

(ii) Let G be the isometry group of the infinite wallpaper pattern, a portion of which is illustrated below. (A copy of the diagram on white paper is provided; if you wish, you may write on it and hand it in with your answer.)



(a) Describe geometrically *all* the translations, reflections and rotations (if any) in G. State clearly the vectors of any translations, lines of any reflections, and the centres and angles of any rotations. Specify two more elements of G that are not a translation, rotation or reflection. ***(8 marks)***

(b) Find a list of four isometries that generate G. Justify your answer.

(10 marks)

(c) Find n and θ such that $\psi(G)$ is equal to $R_\theta D_n R_\theta^{-1}$ (where D_n is the dihedral group with $2n$ elements). Justify your answer. ***(4 marks)***

3 (i) (a) Give the definition of an action of a group G on a set X.
(3 marks)

(b) Given a homomorphism $\phi : G \to S(X)$ explain how to define an action of G on X and prove that it satisfies the necessary axioms.
(4 marks)

(ii) Let G be a group.

(a) Prove that for $a \in G$ the map $l_a : G \to G$ defined by $x \mapsto l_a(x) := ax$ is an element of $S(G)$ and that the map $G \to S(G), a \mapsto l_a$ is a homomorphism. ***(6 marks)***

(b) Show that the corresponding action of G on itself is faithful.
(2 marks)

(iii) (a) Prove that the symmetry group of a regular tetrahedron is isomorphic to S_4. ***(6 marks)***

(b) Determine the symmetry group if one of the faces of the tetrahedron is coloured red and the other three faces are blue (describe the elements of this group and identify the group). ***(4 marks)***

4 (i) State the Sylow theorems. You should carefully define all the terms and notation used. ***(5 marks)***

(ii) Let G be a group of order 2019 with fewer than 10 elements of order 3.

(a) Prove that there is a unique Sylow 3-subgroup P and a unique Sylow 673-subgroup N. ***(4 marks)***

(b) Show that every element of P commutes with every element of N.
(5 marks)

(c) Use this to prove that G is cyclic. ***(5 marks)***

(iii) (a) Give the definition of a simple group. ***(2 marks)***

(b) By considering an appropriate group action prove that there is no simple group of order 160. ***(4 marks)***

End of Question Paper

Diagram for Question 2

Your registration number: ________________