The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2018–2019**

MAS350 Measure and Probability **2 hours 30 minutes**

*Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.*

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student

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1 (i) Let (S, Σ) be a measurable space.

(a) Write down the definition of a *measure* on (S, Σ). ***(3 marks)***

(b) The *Dirac mass* δ_x at a point $x \in S$ is defined by

$$\delta_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Show that δ_x is a measure on (S, Σ). ***(6 marks)***

(ii) Let (S, Σ, m) be a measure space.

(a) If $A, B \in \Sigma$, show that

$$m(A \cup B) + m(A \cap B) = m(A) + m(B).$$

(3 marks)

(b) Use induction to deduce that if $A_1, A_2, \ldots, A_n \in \Sigma$, then

$$m\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} m(A_i).$$

(5 marks)

(c) If (A_n) is a sequence of sets in Σ, show that

$$m\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} m(A_n).$$

(5 marks)

[Hint: Use the fact that if (B_n) is a sequence of sets in Σ, with $B_n \subseteq B_{n+1}$ for all $n \in \mathbb{N}$, then

$$m\left(\bigcup_{n=1}^{\infty} B_n\right) = \lim_{n \to \infty} m(B_n).]$$

(iii) Calculate the Lebesgue measure of the set

$$A = [0, 1] - [0, 1/2] - \bigcup_{n=1}^{\infty} \left(\frac{1}{2} + \frac{1}{3^{n+1}}, \frac{1}{2} + \frac{1}{3^n}\right).$$

(3 marks)

2 Throughout this question (S, Σ) is a measurable space, and $\mathbb{R}$ is equipped with its usual Borel σ–algebra $\mathcal{B}(\mathbb{R})$.

(i) Recall that $f : S \to \mathbb{R}$ is a *measurable function* if $f^{-1}((a, \infty)) \in \Sigma$ for all $a \in \mathbb{R}$. Show that this is equivalent to the requirement that $f^{-1}((-\infty, a]) \in \Sigma$ for all $a \in \mathbb{R}$. ***(4 marks)***

(ii) The *indicator function* $\mathbf{1}_A$ of the set $A \in \Sigma$ is defined by

$$\mathbf{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

(a) Show that $\mathbf{1}_A$ is a measurable function. ***(3 marks)***

(b) Is $\mathbf{1}_A$ measurable if $A \notin \Sigma$? Justify your answer. ***(2 marks)***

(c) If $A, B \in \Sigma$ with $B \subseteq A$, show directly from the definition of indicator function that

$$\mathbf{1}_{A-B} = \mathbf{1}_A - \mathbf{1}_B.$$

(6 marks)

[Hint: There are three cases to consider.]

(iii) Explain why the following functions from $\mathbb{R}$ to $\mathbb{R}$ are measurable:

(a) $f(x) = \sin(2x)$,

(b) $f(x) = \mathbf{1}_{[0,1]}(x)\sin(2x)$,

(c) $f(x) = \sqrt{\sin(2\mathbf{1}_{[0,1]}(x))}$. ***(4 marks)***

(iv) Let $f : \mathbb{R} \to (c, \infty)$ be a measurable function, where $c \geq 0$. Define

$$g(x) = \frac{1}{f(x) - c}$$

for all $x \in \mathbb{R}$. Show that g is measurable. If $h : \mathbb{R} \to \mathbb{R}$ is measurable, deduce that F is a measurable function, where for all $x \in \mathbb{R}$,

$$F(x) = \frac{h(x)}{f(x) - c}.$$

(6 marks)

3 (i) Consider the following functions from $\mathbb{R}$ to $\mathbb{R}$:

$$f(x) = \begin{cases} -4 & \text{for all } -3 \leq x < 1 \\ 5 & \text{for all } 1 \leq x < 4 \\ 0 & \text{otherwise.} \end{cases}$$

$$g(x) = \begin{cases} 3 & \text{for all } -2 \leq x < 0 \\ -2 & \text{for all } 0 \leq x < 3 \\ 1 & \text{for all } 3 \leq x < 5 \\ 0 & \text{otherwise.} \end{cases}$$

(a) Write f and g explicitly as simple functions. Is fg a simple function? If so, write it explicitly. ***(10 marks)***

(b) Calculate $\int_{\mathbb{R}} f(x)g(x)dx$, $\int_{\mathbb{R}} |f(x)g(x)|dx$ and $\int_{\mathbb{R}} (|f(x)g(x)| - f(x)g(x))dx$. ***(5 marks)***

(ii) Let (S, Σ, m) be a measure space, $f : S \to \mathbb{R}$ be an integrable function and $g : S \to \mathbb{R}$ be a bounded function. What further condition must be imposed on g for the function fg to be integrable? Prove explicitly that when your condition is imposed, then $\int_S |f(x)g(x)|dm(x) < \infty$. ***(3 marks)***

(iii) Let (S, Σ, m) be a measure space. State *Markov's inequality* for a non–negative measurable function defined on S, and use it to prove that if $f : S \to \mathbb{R}$ is a measurable function for which $\int_S f^2 dm = 0$,

then $f = 0$ (a.e.). ***(7 marks)***

4 Throughout this question, the half–line $[0, \infty)$ is equipped with Lebesgue measure on its Borel σ–algebra, and $f : [0, \infty) \to \mathbb{R}$ is an integrable function.

(i) Show that the *Laplace transform* $\mathcal{L}(f)$ exists, in that $|\mathcal{L}(f)(u)| < \infty$ for all $u \in [0, \infty)$, where

$$\mathcal{L}(f)(u) = \int_{[0,\infty)} e^{-ux} f(x) dx.$$

Quote any results that you need from the course. ***(4 marks)***

(ii) Obtain an expression for $\mathcal{L}(f)(u)$ when $f = \mathbf{1}_{[a,b]}$ for $0 \leq a < b < \infty$. ***(3 marks)***

(iii) If $f, g : [0, \infty) \to \mathbb{R}$ are integrable functions, and $a, b \in \mathbb{R}$, show that for all $u \in [0, \infty)$,

$$\mathcal{L}(af + bg)(u) = a\mathcal{L}(f)(u) + b\mathcal{L}(f)(u).$$

(3 marks)

(iv) State *Lebesgue's dominated convergence theorem* and use it to prove that the mapping $u \to \mathcal{L}(f)(u)$ is continuous from $[0, \infty)$ to $\mathbb{R}$. ***(7 marks)***

(v) Assuming that $\int_{[0,\infty)} x|f(x)|dx < \infty$, show that the mapping $u \to \mathcal{L}(f)(u)$ is differentiable for all $u \in (0, \infty)$, and that its derivative is given by

$$\mathcal{L}(f)'(u) = -\mathcal{L}(g)(u),$$

where $g(x) = xf(x)$. ***(8 marks)***

[Hint: Use Lebesgue's dominated convergence theorem and the fact that $1 - e^{-y} \leq y$, for all $y \geq 0$.]

5 (i) let (S, Σ, m) be a measure space wherein the measure m is finite.

(a) If $A \in \Sigma$, what can you say about $m(A) + m(A^c)$? ***(2 marks)***

(b) If (A_n) is a sequence of sets in Σ, define the sets $\liminf_{n\to\infty} A_n$ and $\limsup_{n\to\infty} A_n$. ***(2 marks)***

(c) Explain briefly why $\left(\liminf_{n\to\infty} A_n\right)^c = \limsup_{n\to\infty} A_n^c$, and hence deduce that

$$m\left(\limsup_{n\to\infty} A_n^c\right) = M - m\left(\liminf_{n\to\infty} A_n\right),$$

where M is the *total mass* of the measure m. What form does the last identity take when m is a probability measure? ***(4 marks)***

(ii) Let $(\Omega, \mathcal{F}, P)$ be a probability space. State both parts of the *Borel–Cantelli lemma*, and prove the part that requires an independence assumption.

[Hint: Use the inequality $e^{-x} \geq 1 - x$ for $x \geq 0$.] ***(11 marks)***

(iii) Consider a sequence of independent rolls of a fair die. Show that the run 614325 appears infinitely often. ***(6 marks)***

End of Question Paper