



The  
University  
Of  
Sheffield.

**SCHOOL OF MATHEMATICS AND STATISTICS**

**Spring Semester  
2018–2019**

**Stochastic Processes and Financial Mathematics**

**3 hours**

*Candidates should attempt **ALL** questions.*

*The maximum marks for the various parts of the questions are indicated.*

*The paper will be marked out of 100.*

**Please leave this exam paper on your desk  
Do not remove it from the hall**

Registration number from U-Card (9 digits)  
to be completed by student

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1 Let  $\Omega = \{1, 2, 3, 4, 5\}$  and define two subsets

$$E_1 = \{1, 3, 5\}$$

$$E_2 = \{1, 2, 3, 4\}.$$

(a) List all elements of the  $\sigma$ -field  $\mathcal{F}$  generated by  $E_1$  and  $E_2$ . (7 marks)

(b) Is the function  $X : \Omega \rightarrow \mathbb{R}$  defined by

$$X(\omega) = \begin{cases} 0 & \text{if } \omega \text{ is even} \\ 1 & \text{if } \omega \text{ is odd} \end{cases}$$

measurable with respect to  $\mathcal{F}$ ? Justify your answer. (3 marks)

2 Let  $(\Omega, \mathcal{F}, \mathbb{P})$  be a probability space. Let  $\mathcal{G} \subseteq \mathcal{F}$  be a  $\sigma$ -field on  $\Omega$ .

(a) Let  $X$  be a random variable. Which of the following properties of conditional expectation are stated correctly? Justification is not required.

(i)  $\mathbb{E}[1 | \mathcal{G}] = 1$ .

(ii)  $\mathbb{E}[\mathbb{E}[X | \mathcal{G}]] = \mathbb{E}[X]$ .

(iii) If  $X$  is independent of  $\mathcal{G}$  then  $\mathbb{E}[X | \mathcal{G}] = 0$ .

(iv) If  $X \leq Y$  then  $\mathbb{E}[X | \mathcal{G}] \leq \mathbb{E}[Y | \mathcal{G}]$ .

(v) If  $Y \in m\mathcal{G}$  then  $\mathbb{E}[XY | \mathcal{G}] = X\mathbb{E}[Y | \mathcal{G}]$ .

(5 marks)

(b) For each of the properties (i)-(v) that is *not* stated correctly in part (a), state a corrected version of the property. (2 marks)

(c) Suppose that  $X$  and  $Y$  are a pair of random variables such that  $\mathbb{E}[X | \mathcal{G}] = Y$  and  $\mathbb{E}[X^2 | \mathcal{G}] = Y^2$ . Show that

$$\mathbb{E}[(X - Y)^2] = 0.$$

(3 marks)

- 3 This question concerns the binomial model, in discrete time, with two assets, cash and stock.

*A brief summary of the binomial model, and associated notation, can be found on the supplementary formula sheet.*

Suppose that we have  $T = 2$  steps of time, and let the parameters of the model be  $p_u = p_d = 0.5$ ,  $u = 1.2$ ,  $d = 0.8$ ,  $r = 1/11$  and  $s = 100$ .

Consider the contingent claim

$$\Phi(S_T) = \begin{cases} 120 - S_T & \text{if } S_T \leq 120 \\ 0 & \text{otherwise.} \end{cases}$$

Draw a recombining tree of the stock price process at time  $t = 0, 1, 2$ . Annotate your tree to show the arbitrage free price for this contingent claim, at each node, along with a portfolio strategy that hedges  $\Phi(S_T)$ . **(10 marks)**

- 4 Consider the binomial model, as in Question 3. Recall that we write  $S_t$  for the value of a unit of stock at time  $t$ .

(a) Write down the contingent claim  $\Phi(S_T)$  of a European call option, with strike price  $K$  and expiry time  $T$ . **(2 marks)**

(b) A contract specifies that:

At time  $T$ , if the current price of stock is greater than  $K$ , the holder must sell one unit of stock in return for  $K$  units of cash.

Write down the contingent claim  $\Phi(S_T)$  of this contract. **(2 marks)**

(c) Are the values, at time 0, of the contracts in parts (a) and (b) related? If so, how? **(2 marks)**

5 Consider an urn containing two different colours of balls: red and black. Initially the urn contains one red ball and one black ball. Then, for each time  $n = 1, 2, \dots$ , we do the following:

- Draw one ball from the urn. Record its colour and place this ball back into the urn.
  - If the ball drawn was red, add one new red ball and one new black ball into the urn.
  - If the ball drawn was black, add two new black balls into the urn.

Therefore, at time  $n = 0, 1, 2, \dots$  the urn contains  $2n + 2$  balls.

Let  $B_n$  denote the number of red balls in the urn at time  $n$  and let  $(\mathcal{F}_n)$  be the filtration generated by  $(B_n)$ . Let

$$M_n = \frac{B_n}{2n + 2}$$

denote the fraction of red balls in the urn at time  $n$ .

(a) Show that  $\mathbb{E}[M_{n+1} | \mathcal{F}_n] = \frac{2n+3}{2n+4}M_n$  and hence show that  $M_n$  is a supermartingale. **(7 marks)**

(b) Deduce that there exists a random variable  $M_\infty$  such that  $M_n \xrightarrow{a.s.} M_\infty$ . **(2 marks)**

(c) (i) Explain briefly why

$$\mathbb{E}[M_{n+1}] = \frac{(1)(3)(5) \dots (2n + 3)}{(2)(4)(6) \dots (2n + 4)}.$$

(ii) Show that the product of the first  $N$  even numbers is given by

$$(2)(4)(6) \dots (2N) = 2^N N!$$

and hence show that  $(2N)! = 2^N N! \times (1)(3)(5) \dots (2N - 1)$ .

(iii) Show that  $\mathbb{E}[M_n] \rightarrow 0$  as  $n \rightarrow \infty$ .

*Hint: You may use Stirling's approximation  $N! \approx \sqrt{2\pi N} N^N e^{-N}$ .*

(iv) Describe, approximately and in words, the state of the urn when  $n$  is large.

**(7 marks)**

**6** Let  $B_t$  be a standard Brownian motion.

(a) Calculate the stochastic differentials

(i)  $dX_t$  where  $X_t = B_t^2$ .

(ii)  $dY_t$  where  $Y_t = B_t^3$ .

(iii)  $dZ_t$  where  $Z_t = tB_t$ .

*(9 marks)*

(b) Show that  $B_t^2 - t$  is a martingale.

*(3 marks)*

(c) Show that  $B_t^3 - 3tB_t$  is a martingale.

*(4 marks)*

**7** Let  $(B_t)$  be a standard Brownian motion. Let  $(X_t)$  be an Ito process satisfying the stochastic differential equation

$$dX_t = X_t dt + e^{-t} dB_t \tag{*}$$

and with initial value  $X_0 = 1$ .

(a) Write equation (\*) in integral form.

*(2 marks)*

(b) Show that  $f(t) = \mathbb{E}[X_t]$  satisfies the ordinary differential equation

$$f'(t) = f(t)$$

and hence give an explicit formula for  $\mathbb{E}[X_t]$ .

*(5 marks)*

(c) Let  $Y_t = X_t^2$ . Find the stochastic differential  $dY_t$ , and hence show that  $v(t) = \mathbb{E}[X_t^2]$  satisfies the ordinary differential equation

$$v'(t) = 2v(t) + e^{-2t}. \tag{†}$$

*(6 marks)*

(d) Check that equation (†) is solved by  $v(t) = Ae^{2t} - \frac{1}{4}e^{-2t}$  and hence give an explicit formula for  $\text{var}(X_t)$ .

*(3 marks)*

8 This question concerns the Black-Scholes model, in continuous time.

*A brief summary of the Black-Scholes model, and associated notation, can be found on the supplementary formula sheet.*

(a) Recall that, in the risk neutral world  $\mathbb{Q}$ , the stock price  $(S_t)$  satisfies

$$dS_t = rS_t dt + \sigma S_t dB_t.$$

Show that  $M_t = e^{-rt}S_t$  is a martingale in the risk neutral world.

*(5 marks)*

(b) Let  $T > 0$  and consider the contingent claim

$$\Phi(S_T) = 5 + 2S_T.$$

(i) Find the value of this contingent claim at time  $t \in [0, T]$ .

(ii) Is it possible to replicate this contingent claim with a constant portfolio? Justify your answer.

*(5 marks)*

(c) Let  $T > 1$  and consider the contingent claim

$$\Psi(S_T) = \frac{S_T}{S_1}.$$

Let  $\Pi_t$  denote the value of this contingent claim at time  $t$ .

(i) Suppose  $t \geq 1$ . Show that

$$\Pi_t = \frac{S_t}{S_1}. \quad (\ddagger)$$

(ii) Is the expression  $(\ddagger)$  valid when  $t < 1$ ? Why or why not?

*(6 marks)*

**End of Question Paper**

# MAS352/452/6052 – Formula Sheet – Part One

Where not explicitly specified, the notation used matches that within the typed lecture notes.

## Modes of convergence

- $X_n \xrightarrow{d} X \Leftrightarrow \lim_{n \rightarrow \infty} \mathbb{P}[X_n \leq x] = \mathbb{P}[X \leq x]$  whenever  $\mathbb{P}[X \leq x]$  is continuous at  $x \in \mathbb{R}$ .
- $X_n \xrightarrow{\mathbb{P}} X \Leftrightarrow \lim_{n \rightarrow \infty} \mathbb{P}[|X_n - X| > a] = 0$  for every  $a > 0$ .
- $X_n \xrightarrow{a.s.} X \Leftrightarrow \mathbb{P}[X_n \rightarrow X \text{ as } n \rightarrow \infty] = 1$ .
- $X_n \xrightarrow{L^p} X \Leftrightarrow \mathbb{E}[|X_n - X|^p] \rightarrow 0$  as  $n \rightarrow \infty$ .

## The binomial model and the one-period model

The binomial model is parametrized by the deterministic constants  $r$  (discrete interest rate),  $p_u$  and  $p_d$  (probabilities of stock price increase/decrease),  $u$  and  $d$  (factors of stock price increase/decrease), and  $s$  (initial stock price).

The value of  $x$  in cash, held at time  $t$ , will become  $x(1+r)$  at time  $t+1$ .

The value of a unit of stock  $S_t$ , at time  $t$ , satisfies  $S_{t+1} = Z_t S_t$ , where  $\mathbb{P}[Z_t = u] = p_u$  and  $\mathbb{P}[Z_t = d] = p_d$ , with initial value  $S_0 = s$ .

When  $d < 1+r < u$ , the risk-neutral probabilities are given by

$$q_u = \frac{(1+r) - d}{u - d}, \quad q_d = \frac{u - (1+r)}{u - d}.$$

The binomial model has discrete time  $t = 0, 1, 2, \dots, T$ . The case  $T = 1$  is known as the one-period model.

## Conditions for the optional stopping theorem (MAS452/6052 only)

The optional stopping theorem, for a martingale  $M_n$  and a stopping time  $T$ , holds if any one of the following conditions is fulfilled:

- (a)  $T$  is bounded.
- (b)  $M_n$  is bounded and  $\mathbb{P}[T < \infty] = 1$ .
- (c)  $\mathbb{E}[T] < \infty$  and there exists  $c \in \mathbb{R}$  such that  $|M_n - M_{n-1}| \leq c$  for all  $n$ .



## MAS352/452/6052 – Formula Sheet – Part Two

Where not explicitly specified, the notation used matches that within the typed lecture notes.

### The normal distribution

$Z \sim N(\mu, \sigma^2)$  has probability density function  $f_Z(z) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(z-\mu)^2}{2\sigma^2}}$ .

Moments:  $\mathbb{E}[Z] = \mu$ ,  $\mathbb{E}[Z^2] = \sigma^2 + \mu^2$ ,  $\mathbb{E}[e^Z] = e^{\mu + \frac{1}{2}\sigma^2}$ .

### Ito's formula

For an Ito process  $X_t$  with stochastic differential  $dX_t = F_t dt + G_t dB_t$ , and a suitably differentiable function  $f(t, x)$ , it holds that

$$dZ_t = \left\{ \frac{\partial f}{\partial t}(t, X_t) + F_t \frac{\partial f}{\partial x}(t, X_t) + \frac{1}{2} G_t^2 \frac{\partial^2 f}{\partial x^2}(t, X_t) \right\} dt + G_t \frac{\partial f}{\partial x}(t, X_t) dB_t$$

where  $Z_t = f(t, X_t)$ .

### Geometric Brownian motion

For deterministic constants  $\alpha, \sigma \in \mathbb{R}$ , and  $u \in [t, T]$  the solution to the stochastic differential equation  $dX_u = \alpha X_u dt + \sigma X_u dB_u$  satisfies

$$X_T = X_t e^{(\alpha - \frac{1}{2}\sigma^2)(T-t) + \sigma(B_T - B_t)}.$$

### The Feynman-Kac formula

Suppose that  $F(t, x)$ , for  $t \in [0, T]$  and  $x \in \mathbb{R}$ , satisfies

$$\begin{aligned} \frac{\partial F}{\partial t}(t, x) + \alpha(t, x) \frac{\partial F}{\partial x}(t, x) + \frac{1}{2} \beta(t, x)^2 \frac{\partial^2 F}{\partial x^2}(t, x) - rF(t, x) &= 0 \\ F(T, x) &= \Phi(x). \end{aligned}$$

If  $X_u$  satisfies  $dX_u = \alpha(u, X_u) dt + \beta(u, X_u) dB_u$ , then

$$F(t, x) = e^{-r(T-t)} \mathbb{E}_{t,x} [\Phi(X_T)].$$

## The Black-Scholes model

The Black-Scholes model is parametrized by the deterministic constants  $r$  (continuous interest rate),  $\mu$  (stock price drift) and  $\sigma$  (stock price volatility).

The value of a unit of cash  $C_t$  satisfies  $dC_t = rC_t dt$ , with initial value  $C_0 = 1$ .

The value of a unit of stock  $S_t$  satisfies  $dS_t = \mu S_t dt + \sigma S_t dB_t$ , with initial value  $S_0$ .

At time  $t \in [0, T]$ , the price  $F(t, S_t)$  of a contingent claim  $\Phi(S_T)$  (satisfying  $\mathbb{E}^{\mathbb{Q}}[\Phi(S_T)] < \infty$ ) with exercise date  $T > 0$  satisfies the Black-Scholes PDE:

$$\frac{\partial F}{\partial t}(t, s) + rs \frac{\partial F}{\partial s}(t, s) + \frac{1}{2} s^2 \sigma^2 \frac{\partial^2 F}{\partial s^2}(t, s) - rF(t, s) = 0,$$
$$F(T, s) = \Phi(s).$$

The unique solution  $F$  satisfies

$$F(t, S_t) = e^{-r(T-t)} \mathbb{E}^{\mathbb{Q}}[\Phi(S_T) | \mathcal{F}_t]$$

for all  $t \in [0, T]$ . Here, the ‘risk-neutral world’  $\mathbb{Q}$  is the probability measure under which  $S_t$  satisfies

$$dS_t = rS_t dt + \sigma S_t dB_t.$$

## The Gai-Kapadia model of debt contagion (MAS452/6052 only)

A financial network consists of banks and loans, represented respectively as the vertices  $V$  and (directed) edges  $E$  of a graph  $G$ . An edge from vertex  $X$  to vertex  $Y$  represents a loan owed by bank  $X$  to bank  $Y$ .

Each loan has two possible states: healthy, or defaulted. Each bank has two possible states: healthy, or failed. Initially, all banks are assumed to be healthy, and all loans between all banks are assumed to be healthy.

Given a sequence of contagion probabilities  $\eta_j \in [0, 1]$ , we define a model of debt contagion by assuming that:

- (†) For any bank  $X$ , with in-degree  $j$  if, at any point,  $X$  is healthy and one of the loans owed to  $X$  becomes defaulted, then with probability  $\eta_j$  the bank  $X$  fails, independently of all else. All loans owed by bank  $X$  then become defaulted.

Starting from some set of newly defaulted loans, the assumption (†) is applied iteratively until no more loans default.