The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2018–2019**

Applied Probability **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

There are 60 marks available on the paper.

Please leave this exam paper on your desk
Do not remove it from the hall

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1 (a) At a particular location, the states of the weather on a sequence of days are modelled as a discrete time Markov chain with state space $\{W, D\}$, with W representing wet and D representing dry. The transition matrix is assumed to be

$$\begin{pmatrix} 1-\alpha & \alpha \\ \beta & 1-\beta \end{pmatrix},$$

where α and β are unknown values in $(0, 1)$. A sequence of 100 observations from the chain gives the following data on transitions.

Numbers of transitions	Day $t+1$ wet	Day $t+1$ dry
Day t wet	23	17
Day t dry	17	42

(i) Give the likelihood of α and β given this information and assuming the starting state is known. ***(3 marks)***

(ii) Find the maximum likelihood estimates of α and β given this information. ***(4 marks)***

(b) The weather at a different location is also assumed to be modelled by a Markov chain, with a transition matrix of the same form but with possibly different parameters. A sequence of 100 observations at this location gives the following data on transitions.

Numbers of transitions	Day $t+1$ wet	Day $t+1$ dry
Day t wet	14	9
Day t dry	8	68

(i) Find the maximum likelihood estimates of α and β for this location. ***(2 marks)***

(ii) Test the null hypothesis that the transition matrices are the same for the two locations against the alternative that the transition matrices are different. ***(7 marks)***

2 A continuous time Markov chain on state space $\{1, 2, 3\}$ is assumed to have generator matrix

$$\begin{pmatrix} -(1+\alpha) & \alpha & 1 \\ 0 & -1 & 1 \\ 1 & \alpha & -(1+\alpha) \end{pmatrix}$$

for some unknown constant $\alpha > 0$.

(a) Assume that the chain is known to start in state 1 at time 0.

(i) What is the distribution of the time at which the chain first leaves state 1? ***(2 marks)***

(ii) What is the probability that, at the time it first leaves state 1, its move is to state 2? ***(1 mark)***

(iii) Describe the probabilities that the chain is in each of its states at time h for small $h > 0$. You may use Landau o notation. ***(3 marks)***

(b) Find the unique stationary distribution of the chain. ***(4 marks)***

(c) The following table shows transitions observed in the time interval $(0, 5]$, with the chain starting in state 1 at time 0.

Time	State before	State after
0.535	1	2
1.101	2	3
1.232	3	2
3.442	2	3
3.657	3	1
3.754	1	2
3.800	2	3
4.284	3	2

In the above table, the total time within $(0, 5]$ spent in each state is 0.632 in state 1, 3.538 in state 2 and 0.830 in state 3.

(i) Explain why the likelihood of α given these observations can be written as

$$L(\alpha) = \alpha^4 \exp\{-(3.538 + 1.462(1+\alpha))\}.$$

(HINT: Consider the contribution from each holding time, including the final uncompleted holding time, and from each transition between states.) ***(5 marks)***

(ii) Find the maximum likelihood estimate of α. ***(2 marks)***

3 A population of organisms has size N_t at time t. During an interval $(t, t+h]$, where h is small, three things can happen:

- Each of the organisms currently present in the population gives birth with probability $h + o(h)$.
- Each of the organisms currently present in the population dies with probability $\mu h + o(h)$.
- An immigrant organism joins the population with probability $h + o(h)$.

The probability of more than one of these things happening can be assumed to be $o(h)$.

Modelling the evolution of N_t over time as a continuous time Markov chain on state space $\{0, 1, 2, 3, \ldots\}$ with generator matrix G:

(a) Explain why

$$G = \begin{pmatrix} -1 & 1 & 0 & 0 & \cdots \\ \mu & -(2+\mu) & 2 & 0 & \cdots \\ 0 & 2\mu & -(3+2\mu) & 3 & \cdots \\ 0 & 0 & 3\mu & -(4+3\mu) & \cdots \\ \vdots & \vdots & \vdots & \vdots & \ddots \end{pmatrix}.$$

(The rows and columns correspond to the states in the order $0, 1, 2, 3, \ldots$.)

(4 marks)

(b) Assuming that $\boldsymbol{\pi} = \begin{pmatrix} \pi_0 & \pi_1 & \pi_2 & \ldots \end{pmatrix}$ satisfies $\boldsymbol{\pi} G = \mathbf{0}$, give the general form of the equations for $\boldsymbol{\pi}$ and the special case relating π_0 and π_1.

(3 marks)

(c) Show that $\pi_j = C/\mu^j$, where C is a constant, solves the equations you found in (b). Hence find a criterion in terms of μ for a stationary distribution to exist, and in the case where a stationary distribution does exist, give the form of the stationary distribution. ***(7 marks)***

4 The locations of trees of a particular species in a forest are modelled by a spatial Poisson process on the square $S = [0, 1]^2$ with intensity function $\lambda(x, y) = (x + y)$. Let T be the subregion of S defined by $T = [0, 1/2]^2$.

(a) What is the distribution of the number of trees of the species in T?

(3 marks)

(b) Conditional on there being exactly 5 trees of the species in S, what is the probability that none of them are in T? ***(4 marks)***

(c) Explain why the x co-ordinates of the locations of the trees of the species on S form a one-dimensional Poisson process on $[0, 1]$, and identify the rate function. ***(6 marks)***

End of Question Paper

Background material for MAS371 exam

For the purposes of the MAS371 exam, you may assume these results in the somewhat simplified form they are given in this document.

Asymptotic normality of maximum likelihood estimators

Given a vector of unknown parameters $\hat{\boldsymbol{\theta}}$, the maximum likelihood estimator $\hat{\boldsymbol{\theta}}$ has a distribution which is asymptotically (in the large sample limit) Normal with mean vector $\boldsymbol{\theta}_0$ and covariance matrix approximately given by $J(\hat{\boldsymbol{\theta}})^{-1}$. Here $\boldsymbol{\theta}_0$ is the true value and $J(\boldsymbol{\theta})$ is the observed information matrix.

The r, s entry of $J(\boldsymbol{\theta})$ is given by

$$J(\boldsymbol{\theta})_{rs} = -\frac{\partial^2 \ell(\boldsymbol{\theta})}{\partial\theta_r \partial\theta_s},$$

where $\ell(\boldsymbol{\theta})$ is the log likelihood.

Wilks' Theorem

Let $\boldsymbol{\theta}$ be a vector of unknown parameters with $\boldsymbol{\theta} \in \Theta$, where Θ is a p-dimensional set. Let $\boldsymbol{\theta}_0$ be the true value.

Simple hypothesis

Consider the null hypothesis $H_0 : \boldsymbol{\theta}_0 = \boldsymbol{\theta}^*$, where $\boldsymbol{\theta}^*$ is a specified value. Then, under H_0,

$$W = -2(\ell(\boldsymbol{\theta}^*) - \ell(\hat{\boldsymbol{\theta}})),$$

where $\hat{\boldsymbol{\theta}}$ is the maximum likelihood estimator and ℓ is the log likelihood, has an asymptotically (in the large sample limit) χ^2 distribution with p degrees of freedom.

Composite hypothesis

Let Θ_0 be a q-dimensional subset of Θ, and consider the null hypothesis $H_0 : \boldsymbol{\theta}_0 \in \Theta_0$. Then, under H_0,

$$W = -2(\ell(\boldsymbol{\theta}^*) - \ell(\hat{\boldsymbol{\theta}})),$$

where $\hat{\boldsymbol{\theta}}$ is the maximum likelihood estimator and ℓ is the log likelihood, has an asymptotically (in the large sample limit) χ^2 distribution with $p - q$ degrees of freedom.

Table of the q-quantile of the χ^2 distribution with ν degrees of freedom

		ν								
		1	2	3	4	5	6	7	8	9
	0.10	0.02	0.21	0.58	1.06	1.61	2.2	2.83	3.49	4.17
	0.50	0.45	1.39	2.37	3.36	4.35	5.35	6.35	7.34	8.34
q	0.90	2.71	4.61	6.25	7.78	9.24	10.64	12.02	13.36	14.68
	0.95	3.84	5.99	7.81	9.49	11.07	12.59	14.07	15.51	16.92
	0.99	6.63	9.21	11.34	13.28	15.09	16.81	18.48	20.09	21.67
		ν								
		10	20	30	40	50	60	70	80	90
	0.10	4.87	12.44	20.6	29.05	37.69	46.46	55.33	64.28	73.29
	0.50	9.34	19.34	29.34	39.34	49.33	59.33	69.33	79.33	89.33
q	0.90	15.99	28.41	40.26	51.81	63.17	74.4	85.53	96.58	107.57
	0.95	18.31	31.41	43.77	55.76	67.5	79.08	90.53	101.88	113.15
	0.99	23.21	37.57	50.89	63.69	76.15	88.38	100.43	112.33	124.12