The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring semester 2018-2019**

Operations Research **2 Hours**

Attempt all FOUR questions.

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1 (i) Use the two-phase method to find the optimal solution for the following linear programming problem:

$$\max \quad z = -2x_1 - 4x_2$$

subject to $x_1, x_2 \geq 0$ and

$$x_1 + x_2 \leq 5,$$
$$3x_1 + 2x_2 \geq 6.$$

State clearly your final optimal solution.

Hint: not counting the preprocessing step, you need only one simplex iteration in phase 1. ***(12 marks)***

(ii) RentCar is developing a replacement policy for a car over a 4-year planning horizon. At the start of each year, a decision is made as to whether this car should be kept in operation or replaced by another car (the replacement car). The replacement car could also be replaced by the same process. The objective is to minimize the total cost of replacement over the 4-year period.

This car and the replacement cars must be in service for a minimum of 1 year and a maximum of 3 years. The following table provides the replacement cost as a function of the year a car is acquired and the number of years in operation. It is assumed that the car is acquired at the start of year 1.

Car acquired at the start of year	Replacement cost (£) for years in operation		
	1	2	3
1	4000	5400	9800
2	4300	6200	8700
3	4800	7100	-
4	4900	-	-

Some of the entries in the table do not have data because they are outside the 4-year planning horizon.

As an example, if the car is replaced at the start of year 2, and the replacement car is kept in service until the end of year 4 (or equivalently the start of year 5), then the total cost is £4000 + £8700 = £12700. To find the policy to minimize the total cost, the shortest-route model is used to formulate the problem.

(a) Using node i to represent the start of year i ($i = 1, 2, 3, 4, 5$), draw the network that represents the problem. ***(4 marks)***

(b) Find the mixed integer-linear programming model for the problem. **Find the formulation only; do not try to find the numerical solution**. ***(9 marks)***

2 Company M mixes four ingredients and a filler to produce a nutritional powder. The powder contains three nutrients: A, B and C. The nutrient contents (unit: grams per kilogram of the ingredient) and the costs (unit: pounds per kilogram of the ingredient) of each ingredient are given in this table:

	A (g/kg)	B (g/kg)	C (g/kg)	Cost (£/kg)
Ingredient 1	1	8	4	4
Ingredient 2	2	1	5	6
Ingredient 3	7	4	1	8
Ingredient 4	2	6	2	5

The following information is known:

- According to marketing regulations, if company M wants to claim that the powder contains a nutrient, then the amount of that nutrient must be no less than a minimum value. The minimum value is different for different nutrients. In one kilogram of the powder, the minimum value is 5 grams for nutrient A, 7 grams for B, and 4 grams for C.
- Company M intends to claim that the powder contains **at least two nutrients**. They have no preference for which ones.
- Company M plans to mix 5000 kilograms of the powder.
- A fixed set-up cost of £300 is incurred if either ingredient 2 or 4 are used.
- The chemical properties of the ingredients are such that, if both ingredients 1 and 3 are used, then ingredient 2 cannot be used.
- Company M purchases ingredient 1 and 2 from company P. Due to profitability considerations, company P does not accept small orders. Therefore, company M can purchase either no less than 100 kilograms of ingredient 1 alone, or no less than 120 kilograms of ingredient 2 alone, or no less than 290 kilograms of the two ingredients combined.
- Starch is used as the filler. Its nutritional contents and cost can be neglected. There is no separate constraint on how much it should be used.

Let x_1, x_2, x_3, and x_4 be the amounts (in kilograms) of the four ingredients in **1 kilogram** of the powder. Formulate the mixed integer linear programming problem from which one can find the optimal solution to minimise the total cost. **Note: do NOT try to find the numerical solution of the problem; you may need to introduce more variables**. *(25 marks)*

3 You are given the following maximisation problem:

$$\max \quad f(x_1, x_2) = -x_1^2 + 2x_1 + 4x_2 + 4, \quad \text{subject to} \quad 4x_2^2 \leq 1. \tag{1}$$

Consider this as the primal problem and let y be the dual variable corresponding to the constraint.

(i) Write down the KKT conditions for the problem defined in Equation (1), assuming the strong duality condition holds. ***(6 marks)***

(ii) Using the KKT conditions, find the optimal solutions for the primal and dual variables as well as the maximum of $f(x_1, x_2)$. ***(5 marks)***

(iii) Define the Lagrangian dual function $v(y)$ and simplify it as far as you can. ***(9 marks)***

(iv) Write down the dual problem based on the $v(y)$ obtained in Part (iii). ***(2 marks)***

(v) Without assuming that the strong duality condition holds, find the optimal solution for the dual problem and show that it is the same as the one obtained in Part (ii). ***(3 marks)***

4 Fonex is a phone maker that makes two phones. The following information is known:

- The unit profit for phone 1 is £30. It requires one hour of work on assembly line 1 and two hours of work on assembly line 2.
- The unit profit for phone 2 is £20. It requires two hours of work on assembly line 1 and one hour of work on assembly line 2.
- Assembly line 1 can be used up to 40 hours per week. Assembly line 2 can be used up to 50 hours per week. These two conditions provide two constraints for the problem.

We define x_1 and x_2 as the numbers of phone 1 and phone 2 produced each week, respectively, and let x_3 and x_4 be the slack variables corresponding to the constraints on the availability of assembly lines 1 and 2. To maximise the total profit, one can formulate a linear programming problem as follows:

$$\max \quad z = 3x_1 + 2x_2 \qquad (\textbf{One unit of } z \textbf{ is £10})$$

subject to

$$\begin{aligned} x_1 + 2x_2 &\leq 40 \\ 2x_1 + x_2 &\leq 50 \\ x_1, x_2 &\geq 0 \end{aligned}$$

As a simplification, we allow x_1 and x_2 to be non-integer numbers. Solving the problem with the simplex method, we find the following optimal tableau:

	x_1	x_2	x_3	x_4	Solution
z	0	0	$1/3$	$4/3$	80
x_1	1	0	$-1/3$	$2/3$	20
x_2	0	1	$2/3$	$-1/3$	10

(i) Using the information given in the optimal tableau, find the optimal solutions for the decision variables, the cost, and the dual variables. ***(3 marks)***

(ii) Determine the optimality range for the unit profit for phone 1. ***(5 marks)***

(iii) If assembly line 2 can be used up to 55 hours per week, what would be the new optimal solution for x_1 and x_2? ***(3 marks)***

(iv) Fonex is considering making a third phone. The phone needs 2 hours of work on assembly line 1 and 2 hours of work on assembly line 2. Let the unit profit for the phone be £P. It is profitable for Fonex to make the phone only when P is bigger than a minimum value. Find this value. ***(4 marks)***

 Question 4 continued on next page

4 (continued)

(v) A new market survey shows that phone 1 is not as popular as phone 2. Therefore Fonex decides that they should not produce more phone 1 than phone 2. Would the current solution still be optimal? If not, find the new optimal solution. ***(10 marks)***

End of Question Paper