Answer four questions. You are advised not to answer more than four questions: if you do, only your best four will be counted.

1. (a) Given a topological space \( X \), define the set \( \pi_0(X) \). You should include a proof that the relevant equivalence relation is in fact an equivalence relation.

(b) Consider \([0,1]\) as a based space with 0 as the basepoint. For \( n \geq 3 \) we define 
\( X_n = \{ z \in \mathbb{C} \mid z^n \in [0,1] \} \):

\[ X_7 \quad X_8 \quad X_9 \]

(i) For which \( n \) and \( m \) (with \( n, m \geq 3 \)) is \( X_n \) homotopy equivalent to \( X_m \)?

(ii) For which \( n \) and \( m \) (with \( n, m \geq 3 \)) is \( X_n \) homeomorphic to \( X_m \)?

Justify your answers carefully.

(c) Give examples as follows, with justification:

(1) A based space \( W \) with \( |\pi_1(W)| = 8 \).

(2) A space \( X \) with two points \( a, b \in X \) such that \( \pi_1(X, a) \) is not isomorphic to \( \pi_1(X, b) \).

(3) A space \( Y \) such that \( H_0(Y) \simeq H_2(Y) \simeq H_4(Y) \simeq H_6(Y) \simeq \mathbb{Z} \) and all other homology groups are trivial.
2 Are the following true or false? Justify your answers.
   (a) $S^6$ is a Hausdorff space. (4 marks)
   (b) The Klein bottle is a retract of $S^1 \times S^1 \times S^1$. (4 marks)
   (c) There is a connected space $X$ with $\pi_1(X) \simeq \mathbb{Z}/2$ and $H_1(X) \simeq \mathbb{Z}$. (4 marks)
   (d) There is a short exact sequence $\mathbb{Z}/9 \to \mathbb{Z}/99 \to \mathbb{Z}/11$. (4 marks)
   (e) If $K$ is a simplicial complex and $L$ is a subcomplex and $H_3(K) = 0$ then $H_3(L) = 0$. (4 marks)
   (f) If $K$ and $L$ are simplicial complexes and $f: |K| \to |L|$ is a continuous map then there is a simplicial map $s: K \to L$ such that $f$ is homotopic to $|s|$. (5 marks)

3 Let $K$ and $L$ be abstract simplicial complexes.
   (a) Define what is meant by a simplicial map from $K$ to $L$. (3 marks)
   (b) Let $s, t: K \to L$ be simplicial maps. Define what it means for $s$ and $t$ to be directly contiguous. (3 marks)
   (c) Prove that if $s$ and $t$ are directly contiguous, then the resulting maps $|s|, |t|: |K| \to |L|$ are homotopic. (3 marks)
   (d) Prove that if $s$ and $t$ are directly contiguous, then the resulting maps $s_*, t_*: H_*(K) \to H_*(L)$ are the same. (You can prove the main formula just for $n = 3$ rather than general $n$.) (9 marks)
   (e) How many injective simplicial maps are there from $\partial \Delta^2$ to itself? Show that no two of them are directly contiguous. (7 marks)
4 Let $U \xrightarrow{i} V \xrightarrow{p} W$ be a short exact sequence of chain complexes and chain maps.

(a) Define what is meant by saying that the above sequence is short exact. 

(3 marks)

Now recall that a snake for the above sequence is a system $(c, w, v, u, a)$ such that

- $c \in H_n(W)$;
- $w \in Z_n(W)$ is a cycle such that $c = [w]$;
- $v \in V_n$ is an element with $p(v) = w$;
- $u \in Z_{n-1}(U)$ is a cycle with $i(u) = d(v) \in V_{n-1}$;
- $a = [u] \in H_{n-1}(U)$.

(b) Prove that for each $c \in H_n(W)$ there is a snake starting with $c$. (8 marks)

(c) Prove that if two snakes have the same starting point, then they also have the same endpoint. (10 marks)

(d) Suppose that the differential $d: V_{n+1} \to V_n$ is surjective. Show that any snake starting in $H_n(W)$ ends with zero. (4 marks)

5 Consider a simplicial complex $K$ with subcomplexes $L$ and $M$ such that $K = L \cup M$. Use the following notation for the inclusion maps:

$$
\begin{array}{ccc}
L \cap M & \xrightarrow{i} & L \\
\downarrow & & \downarrow \\
M & \xrightarrow{g} & K.
\end{array}
$$

(a) State the Seifert-van Kampen Theorem (in a form applicable to simplicial complexes and subcomplexes as above). (4 marks)

(b) State the Mayer-Vietoris Theorem. (5 marks)

(c) State a theorem about the relationship between $\pi_1$ and $H_1$. (3 marks)

(d) Suppose that $|L|$, $|M|$ and $|L \cap M|$ are all homotopy equivalent to $S^1$. Suppose that the maps $i$ and $j$ both have degree two.

1. Find a presentation for $\pi_1|K|$. (3 marks)

2. Find $H_1(K)$. In particular, you should express each nonzero group as a direct sum of terms like $\mathbb{Z}$ or $\mathbb{Z}/n$. (10 marks)

End of Question Paper