The University Of Sheffield.

**SCHOOL OF MATHEMATICS AND STATISTICS** — **Spring Semester 2018–2019**

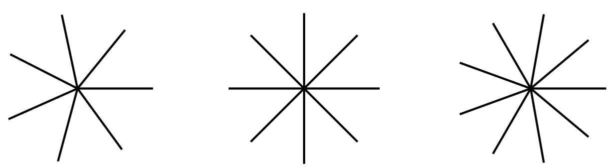
**Algebraic Topology** — **2 hours 30 minutes**

*Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.*

**1** (a) Given a topological space $X$, define the set $\pi_0(X)$. You should include a proof that the relevant equivalence relation is in fact an equivalence relation.

***(8 marks)***

(b) Consider $[0,1]$ as a based space with 0 as the basepoint. For $n \geq 3$ we define $X_n = \{z \in \mathbb{C} \mid z^n \in [0,1]\}$:



$X_7$ $X_8$ $X_9$

(i) For which $n$ and $m$ (with $n, m \geq 3$) is $X_n$ homotopy equivalent to $X_m$?

***(3 marks)***

(ii) For which $n$ and $m$ (with $n, m \geq 3$) is $X_n$ homeomorphic to $X_m$?

***(4 marks)***

Justify your answers carefully.

(c) Give examples as follows, with justification:

(1) A based space $W$ with $|\pi_1(W)| = 8$. ***(3 marks)***

(2) A space $X$ with two points $a, b \in X$ such that $\pi_1(X, a)$ is not isomorphic to $\pi_1(X, b)$. ***(3 marks)***

(3) A space $Y$ such that $H_0(Y) \simeq H_2(Y) \simeq H_4(Y) \simeq H_6(Y) \simeq \mathbb{Z}$ and all other homology groups are trivial. ***(4 marks)***

**2** Are the following true or false? Justify your answers.

(a) $S^5$ is a Hausdorff space. ***(4 marks)***

(b) The Klein bottle is a retract of $S^1 \times S^1 \times S^1$. ***(4 marks)***

(c) There is a connected space $X$ with $\pi_1(X) \simeq \mathbb{Z}/2$ and $H_1(X) \simeq \mathbb{Z}$. ***(4 marks)***

(d) There is a short exact sequence $\mathbb{Z}/9 \to \mathbb{Z}/99 \to \mathbb{Z}/11$. ***(4 marks)***

(e) If $K$ is a simplicial complex and $L$ is a subcomplex and $H_3(K) = 0$ then $H_3(L) = 0$. ***(4 marks)***

(f) If $K$ and $L$ are simplicial complexes and $f\colon |K| \to |L|$ is a continuous map then there is a simplicial map $s\colon K \to L$ such that $f$ is homotopic to $|s|$. ***(5 marks)***

**3** Let $K$ and $L$ be abstract simplicial complexes.

(a) Define what is meant by a *simplicial map* from $K$ to $L$. ***(3 marks)***

(b) Let $s, t\colon K \to L$ be simplicial maps. Define what it means for $s$ and $t$ to be *directly contiguous*. ***(3 marks)***

(c) Prove that if $s$ and $t$ are directly contiguous, then the resulting maps $|s|, |t|\colon |K| \to |L|$ are homotopic. ***(3 marks)***

(d) Prove that if $s$ and $t$ are directly contiguous, then the resulting maps $s_*, t_*\colon H_*(K) \to H_*(L)$ are the same. (You can prove the main formula just for $n = 3$ rather than general $n$.) ***(9 marks)***

(e) How many injective simplicial maps are there from $\partial\Delta^2$ to itself? Show that no two of them are directly contiguous. ***(7 marks)***

**4** Let $U_* \xrightarrow{i} V_* \xrightarrow{p} W_*$ be a short exact sequence of chain complexes and chain maps.

(a) Define what is meant by saying that the above sequence is short exact. ***(3 marks)***

Now recall that a *snake* for the above sequence is a system $(c, w, v, u, a)$ such that

- $c \in H_n(W)$;
- $w \in Z_n(W)$ is a cycle such that $c = [w]$;
- $v \in V_n$ is an element with $p(v) = w$;
- $u \in Z_{n-1}(U)$ is a cycle with $i(u) = d(v) \in V_{n-1}$;
- $a = [u] \in H_{n-1}(U)$.

(b) Prove that for each $c \in H_n(W)$ there is a snake starting with $c$. ***(8 marks)***

(c) Prove that if two snakes have the same starting point, then they also have the same endpoint. ***(10 marks)***

(d) Suppose that the differential $d\colon V_{n+1} \to V_n$ is surjective. Show that any snake starting in $H_n(W)$ ends with zero. ***(4 marks)***

**5** Consider a simplicial complex $K$ with subcomplexes $L$ and $M$ such that $K = L \cup M$. Use the following notation for the inclusion maps:

$$\begin{array}{ccc} L \cap M & \xrightarrow{\;i\;} & L \\ {\scriptstyle j}\big\downarrow & & \big\downarrow{\scriptstyle f} \\ M & \xrightarrow[\;g\;]{} & K. \end{array}$$

(a) State the Seifert-van Kampen Theorem (in a form applicable to simplicial complexes and subcomplexes as above). ***(4 marks)***

(b) State the Mayer-Vietoris Theorem. ***(5 marks)***

(c) State a theorem about the relationship between $\pi_1$ and $H_1$. ***(3 marks)***

(d) Suppose that $|L|$, $|M|$ and $|L \cap M|$ are all homotopy equivalent to $S^1$. Suppose that the maps $i$ and $j$ both have degree two.

(1) Find a presentation for $\pi_1|K|$. ***(3 marks)***

(2) Find $H_*(K)$. In particular, you should express each nonzero group as a direct sum of terms like $\mathbb{Z}$ or $\mathbb{Z}/n$. ***(10 marks)***

**End of Question Paper**