The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Spring Semester 2018–2019

MAS436: Functional Analysis — **2 hours 30 minutes**

Answer ***four*** *questions. If you answer more than four questions, only your best four will be counted.*

Throughout this paper, unless otherwise stated, all vector spaces are either over the field of real numbers, $\mathbb{R}$, or the field of complex numbers, $\mathbb{C}$. We write $\mathbb{K}$ to denote either $\mathbb{R}$ or $\mathbb{C}$.

1 (i) (a) Let $C[0,1]$ be the complex vector space of continuous functions $f\colon [0,1] \to \mathbb{C}$. Prove that we have a norm on $C([0,1])$ defined by the formula

$$\|f\|_0 = \sup\{|f(t)| \mid t \in [0,1]\}.$$

(b) Let $C_0^{(1)}[0,1]$ be the complex vector space of all differentiable functions $f\colon [0,1] \to \mathbb{C}$ with continuous derivative, such that $f(0) = 0$. Prove that we have a norm on $C_0^{(1)}[0,1]$ defined by the formula

$$\|f\|_1 = \|f\|_0 + \|f'\|_0$$

(6 marks)

(ii) What does it mean for a normed vector space to be a *Banach space*. Prove that $C[0,1]$ is a Banach space. ***(9 marks)***

(iii) Let V and W be normed vector spaces. Say what is meant by a *bounded linear map* $T\colon V \to W$. Prove that we have bounded linear maps $D\colon C_0^{(1)}[0,1] \to C[0,1]$ and $I\colon C[0,1] \to C_0^{(1)}[0,1]$ defined by the formulae

$$D(f) = f' \qquad I(f)(x) = \int_0^x f(t)\, dt$$

respectively. ***(5 marks)***

(iv) Is the space $C_0^{(1)}[0,1]$ a Banach space? Justify your answer. ***(5 marks)***

2 (i) State Zorn's lemma, including definition of the terms *maximal* and *upper bound*. ***(4 marks)***

(ii) Let V be a vector space over the field $\mathbb{K}$, and let W be a subspace. Let $f: W \to \mathbb{K}$ be a linear map. Prove that there exists a linear map $F: V \to \mathbb{K}$ such that $F(w) = f(w)$ for all $w \in W$. ***(7 marks)***

(iii) Let V be a normed vector space over the field $\mathbb{K}$. Define the dual space V^* and the norm of a bounded linear map $f: V \to \mathbb{K}$. Prove that the dual space V^* is a normed vector space under this norm. ***(5 marks)***

(iv) State the Hahn-Banach theorem. ***(3 marks)***

(v) Prove that the linear map $\tau: V \to (V^*)^*$ defined by the formula

$$\tau(v)(f) = f(v) \qquad f \in V^*,\ v \in V$$

is an isometry. ***(6 marks)***

3 (i) Let V and W be Banach spaces. State what is meant by a linear map $T: V \to W$ being *open* and having *closed graph*. State the open mapping and closed graph theorems, and use the open mapping theorem to prove the closed graph theorem. ***(9 marks)***

(ii) Let H be a Hilbert space. Let $T: H \to H$ be a linear map. Suppose that $\langle Tu, v\rangle = \langle u, Tv\rangle$ for all $u, v \in H$. Use the closed graph theorem to show that T is continuous. ***(8 marks)***

(iii) Let H be a Hilbert space, and let $T: H \to H$ be a bounded linear map. Define the *adjoint* $T^*: H \to H$. Compute the adjoint of the maps $S, T: L^2(\mathbb{R}) \to L^2(\mathbb{R})$ defined by the formulae

$$S(f)(x) = f(x+1) \qquad T(g)(x) = e^{ix} g(x-1)$$

respectively, where $x \in \mathbb{R}$. ***(8 marks)***

4 (i) Let A be a complex unital Banach algebra, and let $x \in A$. Define the *spectrum* of x. ***(2 marks)***

(ii) Let $x \in A$ satisfy the inequality $\|x\| < 1$. Prove that the element $1 - x$ is invertible. Deduce that if $\lambda \in \text{Spectrum}(x)$ then $|\lambda| < \|x\|$. ***(10 marks)***

(iii) Let H be a Hilbert space, and let $T: H \to H$ be a bounded linear map. Prove that if λ is an eigenvalue of T, then $\lambda \in \text{Spectrum}(T)$. ***(3 marks)***

(iv) Let $R: \ell^2 \to \ell^2$ be the right shift operator. Prove that R has no eigenvalues. Find the spectrum of R. ***(10 marks)***

5 (i) Define what is meant by the statement that a linear map between normed vector spaces is a *compact operator*. ***(2 marks)***

(ii) Let $K: V \to W$ be a compact operator between normed vector spaces V and W. Let (x_n) be a bounded sequence in V. Prove that (Kx_n) has a convergent subsequence. ***(4 marks)***

(iii) Prove that any bounded linear map with finite-dimensional image is compact. You may if you wish use the Heine-Borel theorem without proof. ***(4 marks)***

(iv) Prove that the operator $S: \ell^2 \to \ell^2$ defined by the formula

$$S(a_1, a_2, a_3, \ldots) = (a_1, \frac{a_2}{2^2}, \frac{a_3}{3^2}, \ldots)$$

is compact. You may use without proof here the fact that a norm-limit of a sequence of compact operators is again compact. ***(5 marks)***

(v) What is the definition of a Fredholm operator? ***(2 marks)***

(vi) Define $T: \ell^2 \to \ell^2$ by the formula

$$T(a_1, a_2, a_3, \ldots) = (a_1 + a_3, \frac{a_2}{2^2} + a_4, \frac{a_3}{3^2} + a_5, \ldots)$$

Show that T is Fredholm, and calculate $\operatorname{Index}(T)$. You may use without proof any standard results from the theory of Fredholm operators. ***(8 marks)***

End of Question Paper