1. (i) (a) Let $C[0, 1]$ be the complex vector space of continuous functions $f: [0, 1] \rightarrow \mathbb{C}$. Prove that we have a norm on $C([0, 1])$ defined by the formula

$$\|f\|_0 = \sup\{|f(t)| : t \in [0, 1]\}.$$

(b) Let $C_0^{(1)}[0, 1]$ be the complex vector space of all differentiable functions $f: [0, 1] \rightarrow \mathbb{C}$ with continuous derivative, such that $f(0) = 0$. Prove that we have a norm on $C_0^{(1)}[0, 1]$ defined by the formula

$$\|f\|_1 = \|f\|_0 + \|f\|_0'$$

(6 marks)

(ii) What does it mean for a normed vector space to be a Banach space? Prove that $C[0, 1]$ is a Banach space. (9 marks)

(iii) Let $V$ and $W$ be normed vector spaces. Say what is meant by a bounded linear map $T: V \rightarrow W$. Prove that we have bounded linear maps $D: C_0^{(1)}[0, 1] \rightarrow C[0, 1]$ and $I: C[0, 1] \rightarrow C_0^{(1)}[0, 1]$ defined by the formulae

$$D(f) = f' \quad I(f)(x) = \int_0^x f(t) \, dt$$

respectively. (5 marks)

(iv) Is the space $C_0^{(1)}[0, 1]$ a Banach space? Justify your answer. (5 marks)
2. (i) State Zorn’s lemma, including definition of the terms \textit{maximal} and \textit{upper bound}. (4 marks)

(ii) Let \( V \) be a vector space over the field \( \mathbb{K} \), and let \( W \) be a subspace. Let \( f: W \to \mathbb{K} \) be a linear map. Prove that there exists a linear map \( F: V \to \mathbb{K} \) such that \( F(w) = f(w) \) for all \( w \in W \). (7 marks)

(iii) Let \( V \) be a normed vector space over the field \( \mathbb{K} \). Define the dual space \( V^* \) and the norm of a bounded linear map \( f: V \to \mathbb{K} \). Prove that the dual space \( V^* \) is a normed vector space under this norm. (5 marks)

(iv) State the Hahn-Banach theorem. (3 marks)

(v) Prove that the linear map \( \tau: V \to (V^*)^* \) defined by the formula
\[
\tau(v)(f) = f(v) \quad f \in V^*, \ v \in V
\]
is an isometry. (6 marks)

3. (i) Let \( V \) and \( W \) be Banach spaces. State what is meant by a linear map \( T: V \to W \) being \textit{open} and having \textit{closed graph}. State the open mapping and closed graph theorems, and use the open mapping theorem to prove the closed graph theorem. (9 marks)

(ii) Let \( H \) be a Hilbert space. Let \( T: H \to H \) be a linear map. Suppose that \( \langle Tu, v \rangle = \langle u, Tv \rangle \) for all \( u, v \in H \). Use the closed graph theorem to show that \( T \) is continuous. (8 marks)

(iii) Let \( H \) be a Hilbert space, and let \( T: H \to H \) be a bounded linear map. Define the \textit{adjoint} \( T^*: H \to H \). Compute the adjoint of the maps \( S, T: L^2(\mathbb{R}) \to L^2(\mathbb{R}) \) defined by the formulae
\[
S(f)(x) = f(x + 1) \quad T(g)(x) = e^{ix}g(x - 1)
\]
respectively, where \( x \in \mathbb{R} \). (8 marks)

4. (i) Let \( A \) be a complex unital Banach algebra, and let \( x \in A \). Define the \textit{spectrum} of \( x \). (2 marks)

(ii) Let \( x \in A \) satisfy the inequality \( \|x\| < 1 \). Prove that the element \( 1 - x \) is invertible. Deduce that if \( \lambda \in \text{Spec}(x) \) then \( |\lambda| < \|x\| \). (10 marks)

(iii) Let \( H \) be a Hilbert space, and let \( T: H \to H \) be a bounded linear map. Prove that if \( \lambda \) is an eigenvalue of \( T \), then \( \lambda \in \text{Spec}(T) \). (3 marks)

(iv) Let \( R: \ell^2 \to \ell^2 \) be the right shift operator. Prove that \( R \) has no eigenvalues. Find the spectrum of \( R \). (10 marks)

MAS436

Continued
5 (i) Define what is meant by the statement that a linear map between normed vector spaces is a *compact operator*. (2 marks)

(ii) Let $K: V \to W$ be a compact operator between normed vector spaces $V$ and $W$. Let $(x_n)$ be a bounded sequence in $V$. Prove that $(Kx_n)$ has a convergent subsequence. (4 marks)

(iii) Prove that any bounded linear map with finite-dimensional image is compact. You may if you wish use the Heine-Borel theorem without proof. (4 marks)

(iv) Prove that the operator $S: \ell^2 \to \ell^2$ defined by the formula

$$S(a_1, a_2, a_3, \ldots) = (a_1, \frac{a_2}{2^2}, \frac{a_3}{3^2}, \ldots)$$

is compact. You may use without proof here the fact that a norm-limit of a sequence of compact operators is again compact. (5 marks)

(v) What is the definition of a Fredholm operator? (2 marks)

(vi) Define $T: \ell^2 \to \ell^2$ by the formula

$$T(a_1, a_2, a_3, \ldots) = (a_1 + a_3, \frac{a_2}{2^2} + a_4, \frac{a_3}{3^2} + a_6, \ldots)$$

Show that $T$ is Fredholm, and calculate $\text{Index}(T)$. You may use without proof any standard results from the theory of Fredholm operators. (8 marks)

End of Question Paper