SCHOOL OF MATHEMATICS AND STATISTICS
Optics and Symplectic Geometry

Attempt all the questions. The allocation of marks is shown in brackets.

Through the paper I denotes an identity matrix and \( J \) denotes a matrix of the form
\[
\begin{bmatrix}
0 & I \\
-I & 0
\end{bmatrix}
\]. All matrices have real entries. The standard symplectic form \( \Omega \) on \( \mathbb{R}^{2n} \) is defined by
\[
\Omega(Z, Z') = Q \cdot P' - P \cdot Q',
\]
where \( Z = (Q, P) \) and \( Z' = (Q', P') \) are elements of \( \mathbb{R}^{2n} \).

1. (i) Calculate in detail, using row operations or otherwise, the determinant of the \( 2n \times 2n \) matrix \( J \). \hspace{1cm} (7 marks)

(ii) Define what it means for a \( 2n \times 2n \) matrix \( S \) to be symplectic. \hspace{1cm} (2 marks)

(iii) Prove that the \( 2n \times 2n \) matrix \( J \) is symplectic. \hspace{1cm} (3 marks)

(iv) (a) Let
\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\]
and
\[
\begin{bmatrix}
A' & B' \\
C' & D'
\end{bmatrix}
\]
be \( 2n \times 2n \) matrices in block form, where \( A, B, C, D, A', B', C' \) and \( D' \) denote \( n \times n \) matrices. Write down the product
\[
\begin{bmatrix}
A & B \\
C & D
\end{bmatrix}
\begin{bmatrix}
A' & B' \\
C' & D'
\end{bmatrix}
\]
in block form. \hspace{1cm} (3 marks)

(b) Let \( S = \begin{bmatrix} A & B \\ C & D \end{bmatrix} \) be \( 2n \times 2n \) matrix written in block form. Prove that \( S \) is symplectic if and only if the three equations
\[
A^T C = C^T A, \quad B^T D = D^T B, \quad A^T D - C^T B = I,
\]
hold. \hspace{1cm} (10 marks)
2

(i) Give a definition of a symplectic form on $\mathbb{R}^{2n}$. \hspace{1cm} (5 marks)

(ii) Let $\sigma$ be a symplectic form on $\mathbb{R}^2$. Show that there is a basis $\{v_1, v_2\}$ for $\mathbb{R}^2$ such that when vectors $Z, Z'$ are expressed as $Z = qv_1 + pv_2$, $Z' = q'v_1 + p'v_2$ then

$$\sigma(Z, Z') = qp' - p'q'.$$

\hspace{1cm} (10 marks)

(iii) Describe how the coordinates $(q, p)$ of a light ray are introduced. Illustrate this with a sketch. \hspace{1cm} (4 marks)

(iv) Suppose that there are two vertical axes, $q_1$ and $q_2$, erected at $z_1$ and $z_2$ respectively. Derive the relation between the two coordinates of a ray, $(q_1, p_1)$ and $(q_2, p_2)$ under the assumption that the ray is almost horizontal. Write this relation in matrix form. \hspace{1cm} (6 marks)

3

(i) Consider $\mathbb{R}^{2n}$ with symplectic form $\sigma$. Give the definition of a Lagrangian subspace. Define what it means for two Lagrangian subspaces of $\mathbb{R}^{2n}$ to be transversal. Show that two Lagrangian subspaces, $L$ and $L'$, are transversal if and only if $L \cap L' = \{0\}$.

\hspace{1cm} (5 marks)

(ii) Verify that the following subspace of $(\mathbb{R}^4, \Omega)$

$$L = \text{span}\{(3, 2, -4, 1), (2, 4, 2, -5)\}$$

is Lagrangian. Is it transversal to $\mathbb{R}^2 \times \{0\}$ and $\{0\} \times \mathbb{R}^2$? \hspace{1cm} (9 marks)

(iii) Denote the standard symplectic form on $\mathbb{R}^{2n}$ by $\Omega$, as usual. On $\mathbb{R}^{4n}$ define $\sigma : \mathbb{R}^{4n} \times \mathbb{R}^{4n} \to \mathbb{R}$ by

$$\sigma((X_1, X_2), (Y_1, Y_2)) = \Omega(X_1, Y_1) - \Omega(X_2, Y_2),$$

where $X_1, X_2, Y_1, Y_2 \in \mathbb{R}^{2n}$.

(a) Verify that $\sigma$ is a symplectic form on $\mathbb{R}^{4n}$. \hspace{1cm} (5 marks)

(b) Let $S$ be an invertible $2n \times 2n$ matrix. Show that $S$ is symplectic if and only if the 'graph' of $S$,

$$G = \{(X, SX) \mid X \in \mathbb{R}^{2n}\}$$

is a Lagrangian subspace of $(\mathbb{R}^{4n}, \sigma)$. (You can use without proof that $S \in Sp(2n)$ if and only if $\Omega(X, Y) = \Omega(SX, SY)$ for all $X, Y \in \mathbb{R}^{2n}$. \hspace{1cm} (6 marks)

Continued
Consider propagation of a ray in $\mathbb{R}^3$. In the Cartesian coordinates $q_1, q_2, z$ a light ray crosses the plane $z = z_0$ at point $(q_1, q_2)$, and the plane $z = z'_0$ at point $(q'_1, q'_2)$. The angles between the $q_1$-axis and the ray and between the $q_2$-axis and the ray are equal to $\frac{\pi}{2} - \varphi_1$ and $\frac{\pi}{2} - \varphi_2$ respectively, where $\varphi_1$ and $\varphi_2$ are small. Hence, the unit vector $v$ in the ray direction can be approximated by $(\varphi_1, \varphi_2, 1)$. The diffraction index $n$ is constant.

(a) Show that the old, $(q_1, q_2, p_1, p_2)$, and new, $(q'_1, q'_2, p'_1, p'_2)$, coordinates of the ray, where $p_1 = n\varphi_1$ and $p_2 = n\varphi_2$, are related by

\[
\begin{bmatrix}
q'_1 \\
q'_2 \\
p'_1 \\
p'_2
\end{bmatrix} =
\begin{bmatrix}
1 & 0 & \frac{z'_0 - z_0}{n} & 0 \\
0 & 1 & 0 & \frac{z'_0 - z_0}{n} \\
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
q_1 \\
qu_2 \\
p_1 \\
p_2
\end{bmatrix}.
\]

(*)

(6 marks)

(b) Using the criterion given in part (iv)(b) of question 1 or otherwise prove that the $4 \times 4$ matrix in (*) is symplectic. (2 marks)

(ii) The Fermat principle states that the light propagates between two points, $A$ and $B$, along a path that minimizes the travel time. The speed of light in a medium is $c/n$, where $c$, a constant, is the speed of light in empty space, and $n \geq 1$ is the refraction index of the medium.

(a) You are given that the refraction index is equal to $n$ for $z < 0$, and $n'$ for $z > 0$. Use the Fermat principle to show that the refracted ray is in the plane determined by the incoming ray and the normal to the surface $z = 0$. (13 marks)

(b) The angles between the incoming ray and the $z$-axis and between the refracted ray and the $z$-axis are equal to $\theta$ and $\theta'$. Use the Fermat principle to show that they are related by Snell's law:

\[n \sin \theta = n' \sin \theta'.\] (4 marks)

End of Question Paper