The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2018–2019**

Galois Theory **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets.

Throughout the paper K denotes a subfield of $\mathbb{C}$ which contains $\mathbb{Q}$.
All field extensions are finite.

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Do not remove it from the hall

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1 (i) State, without proof, the Theorem of the Primitive Element (TPE). **(3 marks)**

(ii) Let $K \subset L$ be an extension of fields.

(a) What does it mean for $\theta : L \to L$ to be a K-*automorphism* of L? **(3 marks)**

(b) Let $\alpha \in L$ be such that $f(\alpha) = 0$ for some polynomial $f(x) \in K[x]$. If θ is a K-automorphism of L, show that $\theta(\alpha)$ is a root of $f(x)$ too. **(5 marks)**

(c) Define the *Galois group* $\mathrm{Gal}(L/K)$ of the field extension $K \subset L$, and say what it means in terms of this group, for $K \subset L$ to be a *Galois extension.* Given an equivalent formulation involving splitting fields. **(4 marks)**

(iii) Let $K \subseteq M \subseteq L$ be finite extensions of fields. Suppose that L/K is Galois.

(a) Prove that L/M is Galois. **(3 marks)**

(b) If M/K is Galois, prove that $\varphi(M) \subseteq M$ for all $\varphi \in \mathrm{Gal}(L/K)$. **(5 marks)**

(c) If M/K is Galois, deduce that $\mathrm{Gal}(L/M) \lhd \mathrm{Gal}(L/K)$, and that

$$\mathrm{Gal}(M/K) \cong \frac{\mathrm{Gal}(L/K)}{\mathrm{Gal}(L/M)}.$$

(10 marks)

2 (i) Let K be a field, and let $f \in K[x]$ be a polynomial of degree n.

(a) Define the *Galois group* $\mathrm{Gal}(f/K)$ of f. **(1 mark)**

(b) Show that there is an injective homomorphism

$$\mathrm{Gal}(f/K) \longrightarrow S_n,$$

where S_n denotes the symmetric group on n letters. **(7 marks)**

(c) Deduce that the splitting field of a polynomial of degree n over K has degree at most $n!$ over K. **(2 marks)**

(ii) What are the Galois groups (up to isomorphism) for the following irreducible quartics over $\mathbb{Q}$?

(a) $x^4 - 3$. **(4 marks)**

(b) $x^4 + 1$. **(3 marks)**

(c) $x^4 + x^3 + x^2 + x + 1$. **(2 marks)**

3 Let $f = x^4 - 2x^2 - 6 \in \mathbb{Q}[x]$ and let M denote the splitting field of f over $\mathbb{Q}$. Let $\alpha = \sqrt{1 + \sqrt{7}}$.

(i) Show that the roots of f are $\pm\alpha$, $\pm\dfrac{i\sqrt{6}}{\alpha}$, and deduce that $M = \mathbb{Q}(\alpha, i\sqrt{6})$. **(4 marks)**

(ii) It is given that $[M : \mathbb{Q}] = 8$. Specify the elements of $\mathrm{Gal}(M/\mathbb{Q})$ by giving their effect on each of α and $i\sqrt{6}$, justifying your answer. **(8 marks)**

(iii) Show that there exist automorphisms $\varphi, \psi \in \mathrm{Gal}(M/\mathbb{Q})$ such that φ has order 4, ψ has order 2, and $\mathrm{Gal}(M/\mathbb{Q}) = \langle \varphi, \psi \rangle$. **(5 marks)**

(iv) Write $\psi\varphi\psi^{-1}$ in the form $\varphi^i\psi^j$. To which well-known group is $\mathrm{Gal}(M/\mathbb{Q})$ isomorphic? **(3 marks)**

(v) Write $L = \mathbb{Q}\left(\alpha + \dfrac{i\sqrt{6}}{\alpha}\right)$. Using the Galois correspondence, find $[L : \mathbb{Q}]$. **(5 marks)**

4 (i) Let a, b be two coprime positive integers that are not squares. Let $L = \mathbb{Q}(\sqrt{a}, \sqrt{b})$. Compute the Galois group $\mathrm{Gal}(L/\mathbb{Q})$ and write down the effect of every element on $\sqrt{a}$ and $\sqrt{b}$. **(3 marks)**

(ii) Prove that the Galois group over $\mathbb{Q}$ of one of the polynomials $x^4 + x + \frac{3}{4}$ and $x^4 + x - \frac{3}{4}$ is A_4 and that the other is S_4. **(7 marks)**

[*You may assume that the resolvent cubic for quartics of the form* x^4+ax+b *is given by* $y^3 - 4by - a^2$, *and that both have discriminant* $256b^3 - 27a^4$.]

(iii) Show that $x^5 - 30x + 12$ over $\mathbb{Q}$ is not soluble by radicals by proving that its Galois group of isomorphic to S_5. (**Hint.** You may use the following fact without proof: Any transitive subgroup of S_5 which contains a transposition is equal to S_5.) **(13 marks)**

End of Question Paper