The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Spring Semester 2018–2019

MAS451 Measure and Probability — **2 hours 30 minutes**

Full marks may be obtained by complete answers to three questions. All answers will be marked, but credit will be given only for the best three answers. Total marks 99.

Please leave this exam paper on your desk
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1 (i) Let (S, Σ) be a measurable space.

(a) Write down the definition of a *measure* on (S, Σ). ***(3 marks)***

(b) The *Dirac mass* δ_x at a point $x \in S$ is defined by

$$\delta_x(A) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

Show that δ_x is a measure on (S, Σ). ***(6 marks)***

(ii) Let (S, Σ, m) be a measure space.

(a) If $A, B \in \Sigma$, show that

$$m(A \cup B) + m(A \cap B) = m(A) + m(B).$$

(3 marks)

(b) Use induction to deduce that if $A_1, A_2, \ldots, A_n \in \Sigma$, then

$$m\left(\bigcup_{i=1}^{n} A_i\right) \leq \sum_{i=1}^{n} m(A_i).$$

(5 marks)

(c) If (A_n) is a sequence of sets in Σ, show that

$$m\left(\bigcup_{n=1}^{\infty} A_n\right) \leq \sum_{n=1}^{\infty} m(A_n).$$

(5 marks)

[Hint: Use the fact that if (B_n) is a sequence of sets in Σ, with $B_n \subseteq B_{n+1}$ for all $n \in \mathbb{N}$, then

$$m\left(\bigcup_{n=1}^{\infty} B_n\right) = \lim_{n \to \infty} m(B_n).]$$

(iii) Calculate the Lebesgue measure of the set

$$A = [0,1] - [0, 1/2] - \bigcup_{n=1}^{\infty} \left(\frac{1}{2} + \frac{1}{3^{n+1}}, \frac{1}{2} + \frac{1}{3^n}\right).$$

(3 marks)

 Question 1 continued on next page

1 (continued)

(iv) (a) If (S, Σ, m) is a measure space, what does it mean for the measure m to be σ–*finite*? ***(2 marks)***

(b) Deduce that Lebesgue measure on the real line (with its Borel σ-algebra) is σ–finite. ***(2 marks)***

(c) If (S, Σ, m) is a measure space and $f : S \to \mathbb{R}$ is a non–negative measurable function, it is proved in the course that m_f is a measure on (S, Σ), where for all $A \in \Sigma$,

$$m_f(A) = \int_A f dm.$$

If f is a simple function, and m is σ–finite, show that m_f is σ–finite. ***(4 marks)***

2 Throughout this question (S, Σ) is a measurable space, and $\mathbb{R}$ is equipped with its usual Borel σ–algebra $\mathcal{B}(\mathbb{R})$.

(i) Recall that $f : S \to \mathbb{R}$ is a *measurable function* if $f^{-1}((a, \infty)) \in \Sigma$ for all $a \in \mathbb{R}$. Show that this is equivalent to the requirement that $f^{-1}((-\infty, a]) \in \Sigma$ for all $a \in \mathbb{R}$. ***(4 marks)***

(ii) The *indicator function* $\mathbf{1}_A$ of the set $A \in \Sigma$ is defined by

$$\mathbf{1}_A(x) = \begin{cases} 1 & \text{if } x \in A \\ 0 & \text{if } x \notin A. \end{cases}$$

(a) Show that $\mathbf{1}_A$ is a measurable function. ***(3 marks)***

(b) Is $\mathbf{1}_A$ measurable if $A \notin \Sigma$? Justify your answer. ***(2 marks)***

(c) If $A, B \in \Sigma$ with $B \subseteq A$, show directly from the definition of indicator function that

$$\mathbf{1}_{A-B} = \mathbf{1}_A - \mathbf{1}_B.$$

(6 marks)

[Hint: There are three cases to consider.]

(iii) Explain why the following functions from $\mathbb{R}$ to $\mathbb{R}$ are measurable:

(a) $f(x) = \sin(2x)$,

(b) $f(x) = \mathbf{1}_{[0,1]}(x) + \sin(2x)$,

(c) $f(x) = \mathbf{1}_{[0,1]}(x) \sin(2x)$,

(d) $f(x) = \sqrt{\sin(2\mathbf{1}_{[0,1]}(x))}$. ***(5 marks)***

(iv) Let $f : \mathbb{R} \to (c, \infty)$ be a measurable function, where $c \geq 0$. Define

$$g(x) = \frac{1}{f(x) - c}$$

for all $x \in \mathbb{R}$. Show that g is measurable. If $h : \mathbb{R} \to \mathbb{R}$ is measurable, deduce that F is a measurable function, where for all $x \in \mathbb{R}$,

$$F(x) = \frac{h(x)}{f(x) - c}.$$

(6 marks)

(v) Let (S, Σ, m) be a measure space. State *Markov's inequality* for a non–negative measurable function defined on S, and use it to prove that if $f : S \to \mathbb{R}$ is a measurable function for which $\int_S f^2 dm = 0$, then $f = 0$ (a.e.). ***(7 marks)***

3 Throughout this question, the half–line $[0, \infty)$ is equipped with Lebesgue measure on its Borel σ–algebra, and $f : [0, \infty) \to \mathbb{R}$ is an integrable function.

(i) Show that the *Laplace transform* $\mathcal{L}(f)$ exists, in that $|\mathcal{L}(f)(u)| < \infty$ for all $u \in [0, \infty)$, where

$$\mathcal{L}(f)(u) = \int_{[0,\infty)} e^{-ux} f(x) dx.$$

Quote any results that you need from the course. ***(4 marks)***

(ii) Obtain an expression for $\mathcal{L}(f)(u)$ when $f = \mathbf{1}_{[a,b]}$ for $0 \leq a < b < \infty$. ***(3 marks)***

(iii) If $f, g : [0, \infty) \to \mathbb{R}$ are integrable functions, and $a, b \in \mathbb{R}$, show that for all $u \in [0, \infty)$,

$$\mathcal{L}(af + bg)(u) = a\mathcal{L}(f)(u) + b\mathcal{L}(f)(u).$$

(3 marks)

(iv) State *Lebesgue's dominated convergence theorem* and use it to prove that the mapping $u \to \mathcal{L}(f)(u)$ is continuous from $[0, \infty)$ to $\mathbb{R}$. ***(7 marks)***

(v) Assuming that $\int_{[0,\infty)} x|f(x)|dx < \infty$, show that the mapping $u \to \mathcal{L}(f)(u)$ is differentiable for all $u \in (0, \infty)$, and that its derivative is given by

$$\mathcal{L}(f)'(u) = -\mathcal{L}(g)(u),$$

where $g(x) = xf(x)$. ***(8 marks)***

[Hint: Use Lebesgue's dominated convergence theorem and the fact that $1 - e^{-y} \leq y$, for all $y \geq 0$.]

(vi) Assume that f and g are integrable functions on $[0, \infty)$, and that g is bounded. Define the *convolution* $f * g$ of f with g by

$$(f * g)(x) = \int_{[0,\infty} f(x - y)g(y)dy,$$

for all $x \in \mathbb{R}$.

(a) Show that $|(f * g)(x)| < \infty$ for all $x \in [0, \infty)$. ***(2 marks)***

(b) Show that $f * g$ is integrable. ***(3 marks)***

(c) Prove that the Laplace transform of the convolution is the product of the Laplace transforms, i.e. that for all $u \geq 0$,

$$\mathcal{L}f * g(u) = \mathcal{L}f(u)\mathcal{L}g(u).$$

(3 marks)

[Hint: You may use the fact that the mapping $(x, y) \to f(x - y)g(y)$ from $S \times S$ to $\mathbb{R}$ is measurable.]

4 (i) let (S, Σ, m) be a measure space wherein the measure m is finite.

(a) If $A \in \Sigma$, what can you say about $m(A) + m(A^c)$? ***(2 marks)***

(b) If (A_n) is a sequence of sets in Σ, define the sets $\liminf_{n\to\infty} A_n$ and $\limsup_{n\to\infty} A_n$. ***(2 marks)***

(c) Explain briefly why $\left(\liminf_{n\to\infty} A_n\right)^c = \limsup_{n\to\infty} A_n^c$, and hence deduce that

$$m\left(\limsup_{n\to\infty} A_n^c\right) = M - m\left(\liminf_{n\to\infty} A_n\right),$$

where M is the *total mass* of the measure m. What form does the last identity take when m is a probability measure? ***(4 marks)***

(ii) Let $(\Omega, \mathcal{F}, P)$ be a probability space. State both parts of the *Borel–Cantelli lemma*, and prove the part that requires an independence assumption.

[Hint: Use the inequality $e^{-x} \geq 1 - x$ for $x \geq 0$.] ***(11 marks)***

(iii) Consider a sequence of independent rolls of a fair die. Show that the run 614325 appears infinitely often. ***(6 marks)***

(iv) (a) Let (Y_n) be a sequence of non–negative random variables which are such that

$$P\left(\liminf_{n\to\infty}\{Y_n < 1/n\}\right) = 1.$$

Show that $Y_n \to 0$ (a.s.) ***(4 marks)***

(b) Let (X_n) be a sequence of random variables which are such that there exists $K > 0$ so that for all $n \in \mathbb{N}$,

$$P(|X_n| \geq 1/n) \leq \frac{K}{n^2}.$$

Show that $X_n \to 0$ (a.s.) ***(4 marks)***

End of Question Paper