Candidates should attempt ALL questions.  
The maximum marks for the various parts of the questions are indicated. 
The paper will be marked out of 100.
1. Let $\Omega = \{1, 2, 3, 4, 5\}$ and define two subsets

$E_1 = \{1, 3, 5\}$

$E_2 = \{1, 2, 3, 4\}$.

(a) State the definition of a $\sigma$-field. \hspace{1cm} (3 marks)

(b) List all elements of the $\sigma$-field $\mathcal{F}$ generated by $E_1$ and $E_2$. \hspace{1cm} (5 marks)

(c) Is the function $X : \Omega \to \mathbb{R}$ defined by

$$X(\omega) = \begin{cases} 
0 & \text{if } \omega \text{ is even} \\
1 & \text{if } \omega \text{ is odd}
\end{cases}$$

measurable with respect to $\mathcal{F}$? Justify your answer. \hspace{1cm} (3 marks)

2. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\mathcal{G} \subseteq \mathcal{F}$ be a $\sigma$-field on $\Omega$.

(a) Let $X$ be a random variable. Which of the following properties of conditional expectation are stated correctly? Justification is not required.

(i) $E[1 | \mathcal{G}] = 1$.

(ii) $E[E[X | \mathcal{G}]] = E[X]$.

(iii) If $X$ is independent of $\mathcal{G}$ then $E[X | \mathcal{G}] = 0$.

(iv) If $X \leq Y$ then $E[X | \mathcal{G}] \leq E[Y | \mathcal{G}]$.

(v) If $Y \in m\mathcal{G}$ then $E[XY | \mathcal{G}] = XE[Y | \mathcal{G}]$.

(5 marks)

(b) For each of the properties (i)-(v) that is not stated correctly in part (a), state a corrected version of the property. \hspace{1cm} (2 marks)

(c) Suppose that $X$ and $Y$ are a pair of random variables such that $E[X | \mathcal{G}] = Y$ and $E[X^2 | \mathcal{G}] = Y^2$. Show that

$$E[(X - Y)^2] = 0.$$

(3 marks)
3 This question concerns the binomial model, in discrete time, with two assets, cash and stock.

A brief summary of the binomial model, and associated notation, can be found on the supplementary formula sheet.

Suppose that we have $T = 2$ steps of time, and let the parameters of the model be $p_u = p_d = 0.5$, $u = 1.2$, $d = 0.8$, $r = 1/11$ and $s = 100$.

Consider the contingent claim

$$\Phi(S_T) = \begin{cases} 
120 - S_T & \text{if } S_T \leq 120 \\
0 & \text{otherwise}.
\end{cases}$$

Draw a recombining tree of the stock price process at time $t = 0, 1, 2$. Annotate your tree to show the arbitrage free price for this contingent claim, at each node, along with a portfolio strategy that hedges $\Phi(S_T)$. \hspace{1cm} (10 marks)

4 Consider the binomial model, as in Question 3. Recall that we write $S_t$ for the value of a unit of stock at time $t$.

(a) Write down the contingent claim $\Phi(S_T)$ of a European call option, with strike price $K$ and expiry time $T$. \hspace{1cm} (2 marks)

(b) A contract specifies that:

At time $T$, if the current price of stock is greater than $K$, the holder must sell one unit of stock in return for $K$ units of cash.

Write down the contingent claim $\Phi(S_T)$ of this contract. \hspace{1cm} (2 marks)

(c) Are the values, at time 0, of the contracts in parts (a) and (b) related? If so, how? \hspace{1cm} (2 marks)
5 Consider an urn containing two different colours of balls: red and black. Initially the urn contains one red ball and one black ball. Then, for each time \( n = 1, 2, \ldots \), we do the following:

- Draw one ball from the urn. Record its colour and place this ball back into the urn.
  - If the ball drawn was red, add one new red ball and one new black ball into the urn.
  - If the ball drawn was black, add two new black balls into the urn.

Therefore, at time \( n = 0, 1, 2, \ldots \) the urn contains \( 2n + 2 \) balls.

Let \( B_n \) denote the number of red balls in the urn at time \( n \) and let \( (F_n) \) be the filtration generated by \( (B_n) \). Let

\[
M_n = \frac{B_n}{2n + 2}
\]
denote the fraction of red balls in the urn at time \( n \).

(a) Show that \( \mathbb{E}[M_{n+1} | F_n] = \frac{2n+3}{2n+4} M_n \) and hence show that \( M_n \) is a supermartingale. \( (7 \text{ marks}) \)

(b) Deduce that there exists a random variable \( M_\infty \) such that \( M_n \xrightarrow{a.s.} M_\infty \). \( (2 \text{ marks}) \)

6 Let \( (X_n) \) be a sequence of independent, identically distributed random variables, such that

\[
\mathbb{P}[X_n = 0] = 1 - p, \quad \mathbb{P}[X_n = 1] = p
\]
where \( p \in (0, 1] \). Let

\[
S_n = \sum_{i=1}^{n} X_i
\]
and let \( (F_n) \) be the filtration generated by \( (X_n) \).

(a) Show that

\[
S_n - pn
\]
is a martingale with respect to \( (F_n) \). \( (5 \text{ marks}) \)

(b) State the definition of a stopping time. \( (2 \text{ marks}) \)

(c) Show that \( T = \min\{n; S_n = 10\} \) is a stopping time and find the value of \( \mathbb{E}[T] \). \( (8 \text{ marks}) \)
7 Let \( B_t \) be a standard Brownian motion.

(a) Calculate the stochastic differentials

(i) \( dX_t \) where \( X_t = B_t^2 \).

(ii) \( dY_t \) where \( Y_t = B_t^3 \).

(iii) \( dZ_t \) where \( Z_t = tB_t \).  

(b) Show that \( B_t^2 - t \) is a martingale.

(c) Show that \( B_t^3 - 3tB_t \) is a martingale.  

(9 marks)

(3 marks)

(4 marks)

8 Let \( (B_t) \) be a standard Brownian motion. Let \( (X_t) \) be an Ito process satisfying the stochastic differential equation

\[
    dX_t = X_t \, dt + e^{-t} \, dB_t
\]

and with initial value \( X_0 = 1 \).

(a) Write equation (*) in integral form.  

(b) Show that \( f(t) = \mathbb{E}[X_t] \) satisfies the ordinary differential equation

\[
    f'(t) = f(t)
\]

and hence give an explicit formula for \( \mathbb{E}[X_t] \).  

(2 marks)

(5 marks)
This question concerns the Black-Scholes model, in continuous time.

*A brief summary of the Black-Scholes model, and associated notation, can be found on the supplementary formula sheet.*

(a) Let $T > 0$ and consider the contingent claim

$$\Phi(S_T) = 5 + 2S_T.$$  

(i) Find the value of this contingent claim at time $t \in [0, T]$.

(ii) Is it possible to replicate this contingent claim with a constant portfolio? Justify your answer.  

*(5 marks) *

(b) Let $T > 1$ and consider the contingent claim

$$\Psi(S_T) = \frac{S_T}{S_1}.$$  

Let $\Pi_t$ denote the value of this contingent claim at time $t$.

(i) Suppose $t \geq 1$. Show that

$$\Pi_t = \frac{S_t}{S_1}.$$  

(‡)

(ii) Is the expression (‡) valid when $t < 1$? Why or why not?  

*(6 marks) *

10 Consider the Gai-Kapadia model of debt contagion, on the financial network

![Financial network diagram]

with contagion probabilities $\eta_{ij} = \frac{1}{1+j}$.

*A brief summary of the Gai-Kapadia model, and associated notation, can be found on the supplementary formula sheet.*

Suppose that bank $A$ fails, and defaults on all of its loans. Calculate the probability that the resulting cascade of defaults causes bank $E$ to fail.  

*(5 marks)*

*End of Question Paper*
Where not explicitly specified, the notation used matches that within the typed lecture notes.

Modes of convergence

- \( X_n \xrightarrow{d} X \iff \lim_{n \to \infty} P[X_n \leq x] = P[X \leq x] \) whenever \( P[X \leq x] \) is continuous at \( x \in \mathbb{R} \).
- \( X_n \xrightarrow{p} X \iff \lim_{n \to \infty} P[|X_n - X| > a] = 0 \) for every \( a > 0 \).
- \( X_n \xrightarrow{a.s.} X \iff P[X_n \to X \text{ as } n \to \infty] = 1 \).
- \( X_n \xrightarrow{L_p} X \iff \mathbb{E}[|X_n - X|^p] \to 0 \text{ as } n \to \infty \).

The binomial model and the one-period model

The binomial model is parametrized by the deterministic constants \( r \) (discrete interest rate), \( p_u \) and \( p_d \) (probabilities of stock price increase/decrease), \( u \) and \( d \) (factors of stock price increase/decrease), and \( s \) (initial stock price).

The value of \( x \) in cash, held at time \( t \), will become \( x(1 + r) \) at time \( t + 1 \).

The value of a unit of stock \( S_t \), at time \( t \), satisfies \( S_{t+1} = Z_t S_t \), where \( P[Z_t = u] = p_u \) and \( P[Z_t = d] = p_d \), with initial value \( S_0 = s \).

When \( d < 1 + r < u \), the risk-neutral probabilities are given by

\[
q_u = \frac{(1 + r) - d}{u - d}, \quad q_d = \frac{u - (1 + r)}{u - d}.
\]

The binomial model has discrete time \( t = 0, 1, 2, \ldots, T \). The case \( T = 1 \) is known as the one-period model.

Conditions for the optional stopping theorem (MAS452/6052 only)

The optional stopping theorem, for a martingale \( M_n \) and a stopping time \( T \), holds if any one of the following conditions is fulfilled:

(a) \( T \) is bounded.

(b) \( M_n \) is bounded and \( P[T < \infty] = 1 \).

(c) \( \mathbb{E}[T] < \infty \) and there exists \( c \in \mathbb{R} \) such that \( |M_n - M_{n-1}| \leq c \) for all \( n \).
The normal distribution

$Z \sim N(\mu, \sigma^2)$ has probability density function $f_Z(z) = \frac{1}{\sqrt{2\pi}\sigma}e^{-\frac{(z-\mu)^2}{2\sigma^2}}$.

Moments: $E[Z] = \mu$, $E[Z^2] = \sigma^2 + \mu^2$, $E[e^Z] = e^{\mu + \frac{1}{2}\sigma^2}$.

Ito’s formula

For an Ito process $X_t$ with stochastic differential $dX_t = F_t\,dt + G_t\,dB_t$, and a suitably differentiable function $f(t, x)$, it holds that

$$dZ_t = \left\{ \frac{\partial f}{\partial t}(t, X_t) + F_t\frac{\partial f}{\partial x}(t, X_t) + \frac{1}{2} G_t^2\frac{\partial^2 f}{\partial x^2}(t, X_t) \right\} dt + G_t\frac{\partial f}{\partial x}(t, X_t) dB_t$$

where $Z_t = f(t, X_t)$.

Geometric Brownian motion

For deterministic constants $\alpha, \sigma \in \mathbb{R}$, and $u \in [t, T]$ the solution to the stochastic differential equation $dX_u = \alpha X_u\,dt + \sigma X_u\,dB_u$ satisfies

$$X_T = X_te^{(\alpha - \frac{1}{2}\sigma^2)(T-t)+\sigma(B_T-B_t)}.$$

The Feynman-Kac formula

Suppose that $F(t, x)$, for $t \in [0, T]$ and $x \in \mathbb{R}$, satisfies

$$\frac{\partial F}{\partial t}(t, x) + \alpha(t, x)\frac{\partial F}{\partial x}(t, x) + \frac{1}{2} \beta(t, x)^2\frac{\partial^2 F}{\partial x^2}(t, x) - rF(t, x) = 0$$

$$F(T, x) = \Phi(x).$$

If $X_u$ satisfies $dX_u = \alpha(u, X_u)\,dt + \beta(u, X_u)\,dB_u$, then

$$F(t, x) = e^{-r(T-t)}E_{t,x}\left[ \Phi(X_T) \right].$$
The Black-Scholes model

The Black-Scholes model is parametrized by the deterministic constants $r$ (continuous interest rate), $\mu$ (stock price drift) and $\sigma$ (stock price volatility).

The value of a unit of cash $C_t$ satisfies $dC_t = rC_t \, dt$, with initial value $C_0 = 1$.

The value of a unit of stock $S_t$ satisfies $dS_t = \mu S_t \, dt + \sigma S_t \, dB_t$, with initial value $S_0$.

At time $t \in [0, T]$, the price $F(t, S_t)$ of a contingent claim $\Phi(S_T)$ (satisfying $\mathbb{E}^Q[\Phi(S_T)] < \infty$) with exercise date $T > 0$ satisfies the Black-Scholes PDE:

$$
\frac{\partial F}{\partial t}(t, s) + rs \frac{\partial F}{\partial s}(t, s) + \frac{1}{2} s^2 \sigma^2 \frac{\partial^2 F}{\partial s^2}(t, s) - rF(t, s) = 0,
$$

$$
F(T, s) = \Phi(s).
$$

The unique solution $F$ satisfies

$$
F(t, S_t) = e^{-r(T-t)} \mathbb{E}^Q[\Phi(S_T) | \mathcal{F}_t]
$$

for all $t \in [0, T]$. Here, the ‘risk-neutral world’ $Q$ is the probability measure under which $S_t$ satisfies

$$
dS_t = rS_t \, dt + \sigma S_t \, dB_t.
$$

The Gai-Kapadia model of debt contagion (MAS452/6052 only)

A financial network consists of banks and loans, represented respectively as the vertices $V$ and (directed) edges $E$ of a graph $G$. An edge from vertex $X$ to vertex $Y$ represents a loan owed by bank $X$ to bank $Y$.

Each loan has two possible states: healthy, or defaulted. Each bank has two possible states: healthy, or failed. Initially, all banks are assumed to be healthy, and all loans between all banks are assumed to be healthy.

Given a sequence of contagion probabilities $\eta_j \in [0, 1]$, we define a model of debt contagion by assuming that:

(†) For any bank $X$, with in-degree $j$ if, at any point, $X$ is healthy and one of the loans owed to $X$ becomes defaulted, then with probability $\eta_j$ the bank $X$ fails, independently of all else. All loans owed by bank $X$ then become defaulted.

Starting from some set of newly defaulted loans, the assumption (†) is applied iteratively until no more loans default.