Data provided: Supplementary Formula Sheet

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS

Spring Semester 2018–2019

Stochastic Processes and Financial Mathematics

3 hours

*Candidates should attempt **ALL** questions.*
The maximum marks for the various parts of the questions are indicated.
The paper will be marked out of 100.

Please leave this exam paper on your desk
Do not remove it from the hall

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to be completed by student

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1 Let $\Omega = \{1,2,3,4,5\}$ and define two subsets

$$E_1 = \{1,3,5\}$$
$$E_2 = \{1,2,3,4\}.$$

(a) State the definition of a σ-field. ***(3 marks)***

(b) List all elements of the σ-field $\mathcal{F}$ generated by E_1 and E_2. ***(5 marks)***

(c) Is the function $X : \Omega \to \mathbb{R}$ defined by

$$X(\omega) = \begin{cases} 0 & \text{if } \omega \text{ is even} \\ 1 & \text{if } \omega \text{ is odd} \end{cases}$$

measurable with respect to $\mathcal{F}$? Justify your answer. ***(3 marks)***

2 Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space. Let $\mathcal{G} \subseteq \mathcal{F}$ be a σ-field on Ω.

(a) Let X be a random variable. Which of the following properties of conditional expectation are stated correctly? Justification is not required.

(i) $\mathbb{E}[1 \,|\, \mathcal{G}] = 1$.

(ii) $\mathbb{E}[\mathbb{E}[X \,|\, \mathcal{G}]] = \mathbb{E}[X]$.

(iii) If X is independent of $\mathcal{G}$ then $\mathbb{E}[X \,|\, \mathcal{G}] = 0$.

(iv) If $X \leq Y$ then $\mathbb{E}[X \,|\, \mathcal{G}] \leq \mathbb{E}[Y \,|\, \mathcal{G}]$.

(v) If $Y \in m\mathcal{G}$ then $\mathbb{E}[XY \,|\, \mathcal{G}] = X\mathbb{E}[Y \,|\, \mathcal{G}]$.

(5 marks)

(b) For each of the properties (i)-(v) that is *not* stated correctly in part (a), state a corrected version of the property. ***(2 marks)***

(c) Suppose that X and Y are a pair of random variables such that $\mathbb{E}[X \,|\, \mathcal{G}] = Y$ and $\mathbb{E}[X^2 \,|\, \mathcal{G}] = Y^2$. Show that

$$\mathbb{E}[(X - Y)^2] = 0.$$

(3 marks)

3 This question concerns the binomial model, in discrete time, with two assets, cash and stock.

A brief summary of the binomial model, and associated notation, can be found on the supplementary formula sheet.

Suppose that we have $T = 2$ steps of time, and let the parameters of the model be $p_u = p_d = 0.5$, $u = 1.2$, $d = 0.8$, $r = 1/11$ and $s = 100$.

Consider the contingent claim

$$\Phi(S_T) = \begin{cases} 120 - S_T & \text{if } S_T \leq 120 \\ 0 & \text{otherwise.} \end{cases}$$

Draw a recombining tree of the stock price process at time $t = 0, 1, 2$. Annotate your tree to show the arbitrage free price for this contingent claim, at each node, along with a portfolio strategy that hedges $\Phi(S_T)$. ***(10 marks)***

4 Consider the binomial model, as in Question 3. Recall that we write S_t for the value of a unit of stock at time t.

(a) Write down the contingent claim $\Phi(S_T)$ of a European call option, with strike price K and expiry time T. ***(2 marks)***

(b) A contract specifies that:

> At time T, if the current price of stock is greater than K, the holder must sell one unit of stock in return for K units of cash.

Write down the contingent claim $\Phi(S_T)$ of this contract. ***(2 marks)***

(c) Are the values, at time 0, of the contracts in parts (a) and (b) related? If so, how? ***(2 marks)***

5 Consider an urn containing two different colours of balls: red and black. Initially the urn contains one red ball and one black ball. Then, for each time $n = 1, 2, \ldots$, we do the following:

- Draw one ball from the urn. Record its colour and place this ball back into the urn.
  - If the ball drawn was red, add one new red ball and one new black ball into the urn.
  - If the ball drawn was black, add two new black balls into the urn.

Therefore, at time $n = 0, 1, 2, \ldots$ the urn contains $2n + 2$ balls.

Let B_n denote the number of red balls in the urn at time n and let $(\mathcal{F}_n)$ be the filtration generated by (B_n). Let

$$M_n = \frac{B_n}{2n+2}$$

denote the fraction of red balls in the urn at time n.

(a) Show that $\mathbb{E}[M_{n+1} \,|\, \mathcal{F}_n] = \frac{2n+3}{2n+4} M_n$ and hence show that M_n is a supermartingale. ***(7 marks)***

(b) Deduce that there exists a random variable M_∞ such that $M_n \stackrel{a.s.}{\to} M_\infty$. ***(2 marks)***

6 Let (X_n) be a sequence of independent, identically distributed random variables, such that

$$\mathbb{P}[X_n = 0] = 1 - p, \qquad \mathbb{P}[X_n = 1] = p$$

where $p \in (0, 1]$. Let

$$S_n = \sum_{i=1}^{n} X_i$$

and let $(\mathcal{F}_n)$ be the filtration generated by (X_n).

(a) Show that

$$S_n - pn$$

is a martingale with respect to $(\mathcal{F}_n)$. ***(5 marks)***

(b) State the definition of a stopping time. ***(2 marks)***

(c) Show that $T = \min\{n \,;\, S_n = 10\}$ is a stopping time and find the value of $\mathbb{E}[T]$. ***(8 marks)***

7 Let B_t be a standard Brownian motion.

(a) Calculate the stochastic differentials

(i) dX_t where $X_t = B_t^2$.

(ii) dY_t where $Y_t = B_t^3$.

(iii) dZ_t where $Z_t = tB_t$.

(9 marks)

(b) Show that $B_t^2 - t$ is a martingale. ***(3 marks)***

(c) Show that $B_t^3 - 3tB_t$ is a martingale. ***(4 marks)***

8 Let (B_t) be a standard Brownian motion. Let (X_t) be an Ito process satisfying the stochastic differential equation

$$dX_t = X_t\,dt + e^{-t}\,dB_t \qquad (*)$$

and with initial value $X_0 = 1$.

(a) Write equation $(*)$ in integral form. ***(2 marks)***

(b) Show that $f(t) = \mathbb{E}[X_t]$ satisfies the ordinary differential equation

$$f'(t) = f(t)$$

and hence give an explicit formula for $\mathbb{E}[X_t]$. ***(5 marks)***

9 This question concerns the Black-Scholes model, in continuous time.

A brief summary of the Black-Scholes model, and associated notation, can be found on the supplementary formula sheet.

(a) Let $T > 0$ and consider the contingent claim

$$\Phi(S_T) = 5 + 2S_T.$$

(i) Find the value of this contingent claim at time $t \in [0, T]$.

(ii) Is it possible to replicate this contingent claim with a constant portfolio? Justify your answer.

(5 marks)

(b) Let $T > 1$ and consider the contingent claim

$$\Psi(S_T) = \frac{S_T}{S_1}.$$

Let Π_t denote the value of this contingent claim at time t.

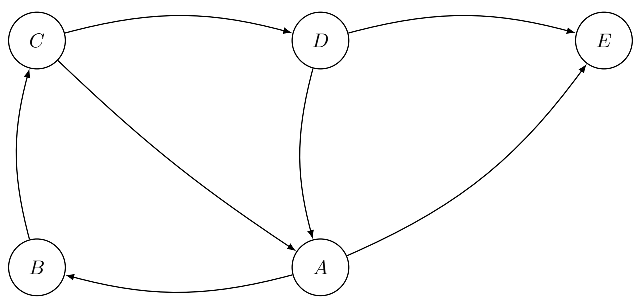
(i) Suppose $t \geq 1$. Show that

$$\Pi_t = \frac{S_t}{S_1}. \quad (\ddagger)$$

(ii) Is the expression ($\ddagger$) valid when $t < 1$? Why or why not?

(6 marks)

10 Consider the Gai-Kapadia model of debt contagion, on the financial network



with contagion probabilities $\eta_j = \frac{1}{1+j}$.

A brief summary of the Gai-Kapadia model, and associated notation, can be found on the supplementary formula sheet.

Suppose that bank A fails, and defaults on all of its loans. Calculate the probability that the resulting cascade of defaults causes bank E to fail. ***(5 marks)***

End of Question Paper

MAS352/452/6052 – Formula Sheet – Part One

Where not explicitly specified, the notation used matches that within the typed lecture notes.

Modes of convergence

- $X_n \stackrel{d}{\to} X \;\Leftrightarrow\; \lim_{n\to\infty} \mathbb{P}[X_n \le x] = \mathbb{P}[X \le x]$ whenever $\mathbb{P}[X \le x]$ is continuous at $x \in \mathbb{R}$.
- $X_n \stackrel{\mathbb{P}}{\to} X \;\Leftrightarrow\; \lim_{n\to\infty} \mathbb{P}[|X_n - X| > a] = 0$ for every $a > 0$.
- $X_n \stackrel{a.s.}{\to} X \;\Leftrightarrow\; \mathbb{P}[X_n \to X \text{ as } n \to \infty] = 1$.
- $X_n \stackrel{L^p}{\to} X \;\Leftrightarrow\; \mathbb{E}[|X_n - X|^p] \to 0$ as $n \to \infty$.

The binomial model and the one-period model

The binomial model is parametrized by the deterministic constants r (discrete interest rate), p_u and p_d (probabilities of stock price increase/decrease), u and d (factors of stock price increase/decrease), and s (initial stock price).

The value of x in cash, held at time t, will become $x(1+r)$ at time $t+1$.

The value of a unit of stock S_t, at time t, satisfies $S_{t+1} = Z_t S_t$, where $\mathbb{P}[Z_t = u] = p_u$ and $\mathbb{P}[Z_t = d] = p_d$, with initial value $S_0 = s$.

When $d < 1 + r < u$, the risk-neutral probabilities are given by

$$q_u = \frac{(1+r) - d}{u - d}, \qquad q_d = \frac{u - (1+r)}{u - d}.$$

The binomial model has discrete time $t = 0, 1, 2, \ldots, T$. The case $T = 1$ is known as the one-period model.

Conditions for the optional stopping theorem (MAS452/6052 only)

The optional stopping theorem, for a martingale M_n and a stopping time T, holds if any one of the following conditions is fulfilled:

(a) T is bounded.

(b) M_n is bounded and $\mathbb{P}[T < \infty] = 1$.

(c) $\mathbb{E}[T] < \infty$ and there exists $c \in \mathbb{R}$ such that $|M_n - M_{n-1}| \le c$ for all n.

MAS352/452/6052 – Formula Sheet – Part Two

Where not explicitly specified, the notation used matches that within the typed lecture notes.

The normal distribution

$Z \sim N(\mu, \sigma^2)$ has probability density function $f_Z(z) = \frac{1}{\sqrt{2\pi}\sigma} e^{-\frac{(z-\mu)^2}{2\sigma^2}}$.

Moments: $\mathbb{E}[Z] = \mu$, $\mathbb{E}[Z^2] = \sigma^2 + \mu^2$, $\mathbb{E}[e^Z] = e^{\mu + \frac{1}{2}\sigma^2}$.

Ito's formula

For an Ito process X_t with stochastic differential $dX_t = F_t\,dt + G_t\,dB_t$, and a suitably differentiable function $f(t, x)$, it holds that

$$dZ_t = \left\{ \frac{\partial f}{\partial t}(t, X_t) + F_t \frac{\partial f}{\partial x}(t, X_t) + \frac{1}{2} G_t^2 \frac{\partial^2 f}{\partial x^2}(t, X_t) \right\} dt + G_t \frac{\partial f}{\partial x}(t, X_t)\,dB_t$$

where $Z_t = f(t, X_t)$.

Geometric Brownian motion

For deterministic constants $\alpha, \sigma \in \mathbb{R}$, and $u \in [t, T]$ the solution to the stochastic differential equation $dX_u = \alpha X_u\,dt + \sigma X_u\,dB_u$ satisfies

$$X_T = X_t e^{(\alpha - \frac{1}{2}\sigma^2)(T-t) + \sigma(B_T - B_t)}.$$

The Feynman-Kac formula

Suppose that $F(t, x)$, for $t \in [0, T]$ and $x \in \mathbb{R}$, satisfies

$$\begin{aligned} \frac{\partial F}{\partial t}(t, x) + \alpha(t, x) \frac{\partial F}{\partial x}(t, x) + \frac{1}{2}\beta(t, x)^2 \frac{\partial^2 F}{\partial x^2}(t, x) - rF(t, x) &= 0 \\ F(T, x) &= \Phi(x). \end{aligned}$$

If X_u satisfies $dX_u = \alpha(u, X_u)\,dt + \beta(u, X_u)\,dB_u$, then

$$F(t, x) = e^{-r(T-t)} \mathbb{E}_{t,x} \left[\Phi(X_T) \right].$$

The Black-Scholes model

The Black-Scholes model is parametrized by the deterministic constants r (continuous interest rate), μ (stock price drift) and σ (stock price volatility).

The value of a unit of cash C_t satisfies $dC_t = rC_t\,dt$, with initial value $C_0 = 1$.

The value of a unit of stock S_t satisfies $dS_t = \mu S_t\,dt + \sigma S_t\,dB_t$, with initial value S_0.

At time $t \in [0,T]$, the price $F(t,S_t)$ of a contingent claim $\Phi(S_T)$ (satisfying $\mathbb{E}^{\mathbb{Q}}[\Phi(S_T)] < \infty$) with exercise date $T > 0$ satisfies the Black-Scholes PDE:

$$\frac{\partial F}{\partial t}(t,s) + rs\frac{\partial F}{\partial s}(t,s) + \frac{1}{2}s^2\sigma^2\frac{\partial^2 F}{\partial s^2}(t,s) - rF(t,s) = 0,$$
$$F(T,s) = \Phi(s).$$

The unique solution F satisfies

$$F(t,S_t) = e^{-r(T-t)}\mathbb{E}^{\mathbb{Q}}[\Phi(S_T)\,|\,\mathcal{F}_t]$$

for all $t \in [0,T]$. Here, the 'risk-neutral world' $\mathbb{Q}$ is the probability measure under which S_t satisfies

$$dS_t = rS_t\,dt + \sigma S_t\,dB_t.$$

The Gai-Kapadia model of debt contagion (MAS452/6052 only)

A financial network consists of banks and loans, represented respectively as the vertices V and (directed) edges E of a graph G. An edge from vertex X to vertex Y represents a loan owed by bank X to bank Y.

Each loan has two possible states: healthy, or defaulted. Each bank has two possible states: healthy, or failed. Initially, all banks are assumed to be healthy, and all loans between all banks are assumed to be healthy.

Given a sequence of contagion probabilities $\eta_j \in [0,1]$, we define a model of debt contagion by assuming that:

(†) For any bank X, with in-degree j if, at any point, X is healthy and one of the loans owed to X becomes defaulted, then with probability η_j the bank X fails, independently of all else. All loans owed by bank X then become defaulted.

Starting from some set of newly defaulted loans, the assumption (†) is applied iteratively until no more loans default.