The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — **Spring Semester 2018-2019**

MAS5051 Probability and Probability Distributions — **2 hours**

RESTRICTED OPEN BOOK EXAMINATION.

Candidates may bring to the examination lecture notes and associated lecture material (including set textbooks) plus a calculator that conforms to University regulations.

Candidates should attempt ***all*** *questions.*

The maximum marks for the various parts of the questions are indicated.

The paper will be marked out of 80.

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student

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1 Let X be a continuous random variable with probability density function given by

$$f_X(x) = \begin{cases} k(x+x^2) & \text{for } 0 \le x \le 2, \\ 0 & \text{otherwise.} \end{cases}$$

(i) Leaving your answers in terms of k, find

(a) $\mathbb{E}(X)$; ***(3 marks)***

(b) The distribution function of X; ***(5 marks)***

(c) $P(X = 0.5)$; ***(1 mark)***

(d) $P(X > 1.5 | X > 1)$. ***(3 marks)***

(ii) Using your answer to part (i)(b), deduce the value of k. ***(2 marks)***

(iii) Let $Y = 3 - 3X$. Find the probability density function of Y. ***(5 marks)***

2 In a group of 100 students, 60 are female and 40 are male. 45 students prefer apples to oranges, of which 27 are female. 40 students, of which 35 are female, are less than 170cm tall. A computer selects a student at random from the group.

(i) What is the probability that the selected student is female? ***(1 mark)***

(ii) Given that the student prefers apples to oranges, what is the probability that the selected student is male? ***(1 mark)***

(iii) Show that the preference for apples or oranges is independent of whether the student is male or female. ***(2 marks)***

(iv) Is the event 'the student is female' independent of the event 'the student is less than 170cm tall'? ***(2 marks)***

(v) Given that the selected student is female, what is the probability that she is less than 170cm tall? ***(2 marks)***

3 Three players play a game 4 times independently. Only one player can win each game; there are no draws. Let X denote the number of times player 1 wins, Y the number of times player 2 wins and Z the number of times player 3 wins. Suppose that, in any single game, player 1 wins with probability 0.5, player 2 wins with probability 0.2 and player 3 wins with probability 0.3.

(i) What is the joint probability distribution of X, Y and Z? ***(1 mark)***

(ii) Calculate the probability $P(X = 2, Y = 1, Z = 1)$. ***(2 marks)***

(iii) Calculate the probability $P(X > 2 | Y = 1)$. ***(4 marks)***

4 Let R be the region defined by $R = \{(x, y) : 0 \leq x \leq y\} \subset \mathbb{R}^2$, and let X and Y be random variables with joint probability density function given by

$$f_{X,Y}(x, y) = \begin{cases} 3e^{-2x-y} & (x, y) \in R, \\ 0 & \text{otherwise.} \end{cases}$$

(i) Verify that $f_{X,Y}(x, y)$ is indeed a valid joint probability density function. ***(4 marks)***

(ii) Find $P(Y \geq 2 \geq X)$. ***(5 marks)***

(iii) Find the marginal probability density functions of X and Y. ***(5 marks)***

(iv) Find the conditional probability density function of Y given $X = x$. ***(3 marks)***

5 Andy and Novak are playing a tennis match. I think Novak is definitely going to win, but my friend is not so sure. I offer to give my friend £10 if Andy wins, as long as my friend gives me £20 if Novak wins. Suppose that the probability that Novak wins is 0.8.

(i) Show that the expected value of my winnings, W, is £14 and the standard deviation of W is £12. ***(4 marks)***

(ii) Suppose that we repeat this bet a total of 25 times, i.e., we bet on each of 25 matches between Andy and Novak. Assuming that Novak's probability of winning remains 0.8 and my winnings from each bet are independent, calculate the mean and standard deviation of my total winnings T, where $T = \sum_{i=1}^{25} W_i$ and W_i is my winnings from the i^{th} bet. ***(3 marks)***

(iii) Use Chebyshev's inequality to give a lower bound for $P(240 \leq T \leq 460)$. ***(4 marks)***

(iv) Use a Normal approximation to calculate $P(240 \leq T \leq 460)$, carefully explaining your answer. You may assume that you do not need to use a continuity correction. You are given the following `R` output to complete the calculation.

```
> pnorm(1.8333)
[1] 0.966621
```

(4 marks)

6 A traffic control engineer believes that the cars passing through a particular intersection arrive at a constant mean rate λ equal to either 2 or 4 every 10 minutes. Prior to collecting any data, the engineer believes that it is much more likely that the rate is 2. In fact, the engineer's prior probabilities are $P(\lambda = 2) = 0.75$ and $P(\lambda = 4) = 0.25$. One day, during a randomly selected 10 minute interval, the engineer observes $x = 6$ cars pass through the intersection.

(i) Explain why the number of cars that pass through the intersection during a randomly selected 10 minute interval, X, might be assumed to be a Poisson distribution with parameter λ. ***(3 marks)***

(ii) Calculate the likelihoods that $\lambda = 2$ and $\lambda = 4$ based on the observation that $x = 6$. ***(3 marks)***

(iii) Calculate the unconditional probability that 6 cars pass through the intersection during a randomly selected 10 minute interval, i.e., calculate $P(X = 6)$. ***(3 marks)***

(iv) In light of the observation that $x = 6$, calculate the posterior probabilities that $\lambda = 2$ and $\lambda = 4$, and hence discuss how the traffic control engineer's beliefs have been altered by the observation. ***(5 marks)***

End of Question Paper