Linear Models

Marks will be awarded for your best five answers.

RESTRICTED OPEN BOOK EXAMINATION
Candidates may bring to the examination lecture notes and associated lecture material (but no textbooks) plus a calculator that conforms to University regulations. There are 100 marks available on the paper.

Please leave this exam paper on your desk
Do not remove it from the hall
Registration number from U-Card (9 digits) to be completed by student

_____ _____ _____ _____ _____ _____
A data set, weight.data in R, on 30 chicks of a particular species of bird at age 2 months gives weight, the weight of the chick in grams, temp, the average temperature in Celsius over the 2 month period, and rain, the total rainfall in millimetres over the 2 month period.

(a) A linear model was fitted to the data in R using the command

\[
weight.lm <- lm(weight~temp+rain,data=weight.data)
\]

giving output

Coefficients:

<p>| | |</p>
<table>
<thead>
<tr>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>(Intercept)</td>
<td>-1120.782</td>
</tr>
<tr>
<td>temp</td>
<td>157.857</td>
</tr>
<tr>
<td>rain</td>
<td>-4.698</td>
</tr>
</tbody>
</table>

The first observation had average temperature 11.7°C and rainfall 15.5mm, and the observed weight of the chick was 435g. Calculate the fitted value and the residual for this observation. 

(2 marks)

(b) Following on from (a), the command

\[
anova(weight.lm)
\]

was entered into R, producing the following output:

Analysis of Variance Table

<table>
<thead>
<tr>
<th></th>
<th>Df</th>
<th>Sum Sq</th>
<th>Mean Sq</th>
<th>F value</th>
<th>Pr(&gt;F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>temp</td>
<td>1</td>
<td>4788381</td>
<td>4788381</td>
<td>19.300</td>
<td>0.0001553 ***</td>
</tr>
<tr>
<td>rain</td>
<td>1</td>
<td>1003809</td>
<td>1003809</td>
<td>4.046</td>
<td>0.0543566 .</td>
</tr>
<tr>
<td>Residuals</td>
<td>27</td>
<td>6698742</td>
<td>248102</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

Based on this output,

(i) assess the evidence for the inclusion of temperature in the model against the null hypothesis where neither temperature nor rainfall is included; 

(2 marks)

(ii) assess the evidence for the inclusion of rainfall in the model given that temperature is already included. 

(2 marks)

(c) Following on from (a) and (b), the code

\[
par(mfrow=c(2,2))
\]

plot(weight.lm$resid,xlab="index",ylab="residuals",main="Index plot")

qqnorm(weight.lm$resid,main="QQ - plot")

hist(weight.lm$resid,xlab="Residuals",main="Histogram")

plot(weight.lm$fit,weight.lm$resid,xlab="Fitted values",ylab="Residuals",main="Residuals versus fitted values")

was entered into R, producing the plots in Figure 1. Using the residual plots, comment on whether there appear to be any problems with the assumptions made for a linear model. 

(4 marks)
1 (continued)

(d) A new model was fitted in R using the command

```r
weight.log.lm <- lm(log(weight) ~ temp + rain, data = weight.data)
```

(i) The output from the new model is

Call:
`lm(formula = log(weight) ~ temp + rain, data = weight.data)`

Coefficients:

<table>
<thead>
<tr>
<th></th>
<th>temp</th>
<th>rain</th>
</tr>
</thead>
<tbody>
<tr>
<td>(Intercept)</td>
<td>1.81651</td>
<td>-0.01282</td>
</tr>
<tr>
<td>temp</td>
<td>0.36827</td>
<td></td>
</tr>
</tbody>
</table>

For the first observation of the data given in (a), calculate the fitted value for the natural log of the weight in the new model and the corresponding residual. (3 marks)

(ii) Residual plots for the new model are shown in Figure 2. Compare these plots with the ones for the original model. Does this model appear to be an improvement? You should give a reason for your answer. (4 marks)
Figure 2: Residual plots for the transformed bird weight data.

(iii) The code

```r
anova(weight.log.lm)
```
gave the output

Analysis of Variance Table

<table>
<thead>
<tr>
<th>Df</th>
<th>Sum Sq</th>
<th>Mean Sq</th>
<th>F value</th>
<th>Pr(&gt;F)</th>
</tr>
</thead>
<tbody>
<tr>
<td>temp</td>
<td>1</td>
<td>26.8556</td>
<td>26.8556</td>
<td>54.189</td>
</tr>
<tr>
<td>rain</td>
<td>1</td>
<td>7.4756</td>
<td>7.4756</td>
<td>15.084</td>
</tr>
<tr>
<td>Residuals</td>
<td>27</td>
<td>13.3809</td>
<td>0.4956</td>
<td>0.4956</td>
</tr>
</tbody>
</table>

Comparing this to the output from (b), how would your answers for (b)(i) and (b)(ii) change with this output? **(3 marks)**
A model for data $y_i$ is that each observation $y_i$ is an observation from a Negative Binomial distribution with parameters $2$ and $p_i$ so that its probability mass function is

$$f(y_i; p_i) = (y_i - 1)p_i^2(1 - p_i)^{y_i-2}.$$ 

(a) Show that the probability mass function can be put into the standard form for a Generalized Linear Model,

$$f(y_i; \theta_i, \phi) = \exp \left\{ w_i y_i \theta_i - b(\theta_i) + c(y_i, \phi) \right\},$$

with $w_i = \phi = 1$, and identify the functions $b$ and $c$. You should explain the relationship between the quantities $\theta_i$ and $p_i$. (8 marks)

(b) (i) Use the function $b$ to show that the mean of $y_i$ is $\frac{2}{1 - e^{\theta_i}}$. (2 marks)

(ii) Give the variance of the distribution of the $y_i$ in terms of $\theta_i$. (2 marks)

(iii) Hence also give the mean and variance of $y_i$ in terms of $p_i$. (2 marks)

(c) What would the canonical link function be for this Generalized Linear Model? (3 marks)

(d) Suppose that the linear predictor is of the form $\eta_i = \beta_0 + \beta_1 x_i + \beta_2 z_i$, for known vectors of explanatory variables $x$ and $z$. Assuming that the canonical link is used, find the estimated value of $p_i$ for an observation with $x_i = 1$ and $z_i = 2$ for a model with $\beta_0 = -8$ and $\beta_1 = \beta_2 = 1$. (3 marks)
An agricultural experiment was carried out to investigate the success of a type of crop, testing the effects of amount of fertilizer and whether the seeds were fresh or not. For each combination of amount of fertilizer and seed freshness, an area of soil was prepared with that amount of fertilizer applied per square metre, 20 seeds were sown, and the number which produced fruit was recorded.

The data are in the following table, and were stored in R as `crop.data` with `fert` being the amount of fertilizer per square metre, `storage` indicating whether the seeds had been stored for a year (coded as 1) or not (coded as 0), `prop.fruit` indicating the proportion out of the 20 which produced fruit, and `n` being 20 for all observations.

<table>
<thead>
<tr>
<th></th>
<th>Amount of fertilizer applied (g m(^{-2}))</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>0</td>
</tr>
<tr>
<td>Fresh</td>
<td>12</td>
</tr>
<tr>
<td>Stored for one year</td>
<td>4</td>
</tr>
</tbody>
</table>

Two Generalized Linear Models with Binomial family and logit link were fitted using the following code.

```r
model.fert <- glm(prop.fruit~fert,family=binomial,weights=n,data=crop.data)
model.both <- glm(prop.fruit~fert+factor(storage),family=binomial,weights=n,data=crop.data)
```

Some R commands and some of the output they produced follow.

```r
> summary(model.fert)

Coefficients:
            Estimate Std. Error z value Pr(>|z|)  
(Intercept) -0.602470  0.245178  -2.457  0.014 *  
fert         0.028487  0.004847   5.878 4.16e-09 ***

Null deviance: 62.558 on 11 degrees of freedom  
Residual deviance: 20.693 on 10 degrees of freedom  
AIC: 59.331

> summary(model.both)

Coefficients:
            Estimate Std. Error z value Pr(>|z|)  
(Intercept) -0.06123  0.28869  -0.212  0.832029  
fert         0.03046  0.00510   5.972 2.34e-09 ***  
factor(storage)1 -1.16689  0.31888  -3.659  0.000253 ***

Null deviance: 62.5585 on 11 degrees of freedom  
Residual deviance: 6.4157 on 9 degrees of freedom  
AIC: 47.054
```

(a) Using the output above, assess the evidence for
(i) whether the model with both storage and fertilizer is an improvement on the null model;  

(ii) whether storage is needed in the model if the model already includes amount of fertilizer.  

(b) In the fitted model with both storage and fertilizer:

(i) Give the odds ratio for success for seeds which have been stored for a year compared with those which are fresh.  

(ii) Give an approximate 95% confidence interval for the odds ratio you calculated in (b)(i).  

(c) In the fitted model with both storage and fertilizer:

(i) What is the estimated probability that a seed will be successful if it is fresh and 50g per square metre of the fertilizer is used?  

(ii) What is the estimated probability that a seed will be successful if it has been stored for a year and 200g per square metre of the fertilizer is used?  

(iii) Do you think your answers to (c)(i) and (c)(ii) would give good predictions if these experiments were actually carried out? Give reasons for your answers.  

(d) If the model given by model both had given a poor fit, what suggestions might you have to find a better model for these data? Give two suggestions.  


A survey is carried out to determine the vitamin C concentration in a specific variety of apple. Ten apple trees are selected at random from an orchard, and then three apples are selected from each tree. Two sample pieces of each apple are then selected at random, and the vitamin C concentration in each piece is then measured in a laboratory. The structure of the dataset is as follows:

```r
> str(appledata)
'data.frame': 60 obs. of 3 variables:
$ vitC : num 1.203 1 0.35 0.371 -0.296 ...
$ tree : Factor w/ 10 levels "1","2","3","4",..: 1 1 1 1 1 1 2 2 2 2 ...
$ apple: Factor w/ 3 levels "1","2","3": 1 1 2 2 3 3 1 1 2 2 ...
```

The following R command was used to fit a mixed effect model (model 1) to the data.

```r
> library(lme4)
> model1 <- lmer(vitC ~ 1 + (1| tree/apple), data=appledata)
```

(a) (i) Let $V_{ijk}$ be the vitamin C concentration measured in piece $k$ for apple $j$ for tree $i$. Write down the algebraic specification of the model that has been fitted to the data making sure you give the distribution of any random effects.

(4 marks)
(ii) The summary of the fitted model (model 1) is given below. Use the R output to give the estimated values for each parameter in your answer to part (i).

```r
> summary(model1)
Linear mixed model fit by REML ['lmerMod']
Formula: vitC ~ 1 + (1 | tree/apple)
  Data: appledata

REML criterion at convergence: 103

Scaled residuals:
       Min      1Q  Median      3Q     Max
-1.80846 -0.53225 -0.03586  0.48503  1.87329

Random effects:
  Groups   Name    Variance  Std.Dev.
  apple:tree (Intercept) 1.098e+00  1.04774
     tree    (Intercept) 1.417e+02  11.90539
  Residual          7.933e-03  0.08907
Number of obs: 60, groups: apple:tree, 30; tree, 10

Fixed effects:
  Estimate Std. Error  t value
 (Intercept) 18.80   REMOVED  4.987

(2 marks)
```

(b) A simpler model (model 2) is fitted using the command

```r
> model2 <- lmer(vitC~ 1 + (1| tree), data=appledata, REML=F)

> logLik(model1)
'log Lik.' -53.70089 (df=4)
> logLik(model2)
'log Lik.' -115.3253 (df=3)
```

Using the R output below, conduct a hypothesis test to choose between model 1 and model 2. Justify your approach.

(4 marks)
(c) Interest lies in the average vitamin C content for the population of apples from this particular type of tree. The standard error of this estimate has been removed from the R output above. Compute the standard error (for model 1) using the variance estimates provided.

**Hint:** The estimate of the fixed effect is given by the sample mean of the data.

*(7 marks)*

(d) Briefly describe what diagnostic checks you could use to validate this mixed effect model.

*(3 marks)*
The University are interested in whether the number of emails received by different lecturers varies between the different faculties. They collect data from $n$ lecturers on the number of emails received each day. Let $x = (x_1, \ldots, x_n)^\top$ be the recorded number of emails received by each lecturer.

A simple statistical model would be that the number of emails received has a Poisson distribution. However, there is reason to believe that lecturers in Arts-based subjects receive emails at a different rate to lecturers in Science-based subjects, and so we will assume that

$$x_i \sim \begin{cases} \text{Poisson}(\lambda) & \text{for Arts subject lecturers} \\ \text{Poisson}(\mu) & \text{for Science lecturers.} \end{cases}$$

Unfortunately, the information about which faculty each lecturer worked in was lost. Let $0 \leq w \leq 1$ be the unknown proportion of Science-based lecturers in the data set. We may thus assume that the $x_i$ are samples from the following mixture distribution:

$$x_i \sim \begin{cases} \text{Poisson}(\lambda) & \text{with probability } 1 - w, \\ \text{Poisson}(\mu) & \text{with probability } w. \end{cases}$$

The corresponding ‘missing’ variables $Y = \{Y_1, \ldots, Y_n\}$ are defined as follows:

$$Y_i = \begin{cases} 0 & \text{if lecturer } i \text{ is in the Arts faculty,} \\ 1 & \text{if lecturer } i \text{ is in the Science faculty.} \end{cases}$$

Define $\theta = (w, \mu, \lambda)$.

(i) Show that the log-likelihood of $\theta$ given the complete data $(x, Y)$ is

$$l(\theta; x, Y) = -\mu \sum Y_i - \lambda \sum (1 - Y_i) + \log \lambda \sum (1 - Y_i)x_i + \log \mu \sum Y_i x_i$$

$$- \sum \log x_i! + \log w \sum Y_i + \log(1 - w) \sum (1 - Y_i)$$

where all sums are over $i = 1, \ldots, n$.

Hint: the probability mass function of a Poisson($\lambda$) random variable is

$$P(X = x) = \frac{e^{-\lambda} \lambda^x}{x!} \quad \text{for } x = 0, 1, 2, \ldots$$

(5 marks)

(ii) Using Bayes’ theorem, show that

$$E(Y_i|x_i, \theta) = \frac{w \mu^x e^{-\mu}}{w \mu^x e^{-\mu} + (1 - w) \lambda^x e^{-\lambda}}.$$

Denote this quantity by $p_i$.

(4 marks)
(iii) The EM algorithm is to be used to obtain the maximum likelihood estimator \( \hat{\theta} = (\hat{w}, \hat{\mu}, \hat{\lambda}) \) of \( \theta \), given the data \( x \). Let the estimate of \( \hat{\theta} \) after \( m \) iterations of the EM algorithm be denoted \( \theta^{(m)} \). By maximising
\[
Q(\theta | \theta^{(m)}) = \mathbb{E}[l(\theta; x, Y)|x, \theta^{(m)}],
\]
show that the updated estimates of \( \hat{w}, \hat{\mu}, \hat{\lambda} \) are
\[
\begin{align*}
\hat{w}^{(m+1)} &= \frac{\sum_{i=1}^{n} p_i}{n} , \\
\hat{\lambda}^{(m+1)} &= \frac{\sum_{i=1}^{n} x_i (1 - p_i)}{\sum_{i=1}^{n} (1 - p_i)} , \\
\hat{\mu}^{(m+1)} &= \frac{\sum_{i=1}^{n} x_i p_i}{\sum_{i=1}^{n} p_i},
\end{align*}
\]
where \( p_i = \mathbb{E}(Y_i|x_i, \theta) \) is your expression derived in part (ii). (5 marks)

(iv) Give an intuitive explanation of the three updates in part (iii). (3 marks)

(b) A survey of voting intentions is conducted. Each person is asked for the political party they intend to vote for in the next general election, the city they live in, and their yearly income. Some values are missing for each covariate, but for different reasons in each case.

In each of the following cases, say whether the data are missing completely at random (MCAR), missing at random (MAR), or not missing at random (NMAR):

(i) Some respondents refuse to say who they will vote for. The person conducting the interview suspects that people who intend to vote for far right parties are less likely to provide this information than the general population.

(ii) Due to a corrupted file, the home city of every fourth person in the survey is lost.

(iii) Some respondents refuse to reveal their income. It is known that older people are less likely than the general population to reveal their income, regardless of what their income is. (3 marks)
Consider a dataset on the record winning times of 35 hill races in Scotland. The covariates are

- **dist** - distance in miles
- **climb** - total height gained during the route, in feet.
- **time** - record time in minutes

The output below shows the structure of the dataset.

```r
> str(hill)
'data.frame': 35 obs. of 3 variables:
$ dist : num NA 6 6 7.5 8 8 16 NA 5 NA ...
$ climb: int 650 2500 NA 800 3070 NA 7500 800 800 650 ...
$ time : num 16.1 NA 33.6 45.6 62.3 ...
>
> head(hill)
      dist climb  time
Greenmantle  NA 650 16.083
Carnethy 6.0 2500  NA
Craig Dunain 6.0  NA 33.650
Ben Rha 7.5  800 45.600
Ben Lomond 8.0 3070 62.267
Goatfell 8.0  NA 73.217
```

(i) The following R command is used.

```r
> hill.mice <- mice(hill, m=5, method=c('norm', 'norm', 'mean'))
```

Describe in detail the statistical procedure that is used to fill in the missing values.

*6 marks*
(ii) Interest lies in the coefficient of $\text{dist} (\beta_1)$ in the linear regression model
\[ \text{time} = \beta_0 + \beta_1 \text{dist} + \beta_2 \text{climb} + \epsilon. \]

Use the R output below to calculate an expected value of $\beta_1$ and its standard error.

\begin{verbatim}
> fit.mice <- with(hill.mice, lm(time ~ dist+climb))
> (coefs = sapply(fit.mice$analyses, coef))

(Intercept) -3.12 1.27 -1.14 -3.47 -2.29
dist 6.46 6.15 6.50 6.16 6.41
climb 0.01 0.01 0.01 0.01 0.01

> apply(coefs, 1, var)
(Intercept) dist climb
3.7e+00 2.9e-02 6.9e-08

> sapply(fit.mice$analyses, function(x) vcov(x)[2,2])
[1] 0.49 0.56 0.47 0.52 0.37
\end{verbatim}

(5 marks)

\( (b) \quad \) A longitudinal study is conducted on the weight gain of rats in the first two weeks of life. Let $y_{ij}$ be the weight of the $i^{th}$ rat at the $j^{th}$ measurement where $i = 1, \ldots, 5$, and $j = 1, \ldots, 3$. Let $x_j$ be the age of the rat in days at the $j^{th}$ measurement with $x_1 = 1, x_2 = 7, x_3 = 14$. We will use the model
\[ y_{ij} = \alpha + a_i + (\beta + b_i)x_j + \epsilon_{ij} \]

where $a_i \sim N(0, \sigma_a^2), b_i \sim N(0, \sigma_b^2)$ and $\epsilon_{ij} \sim N(0, \sigma^2)$, with all random effects independent of each other.
(i) Write the model in matrix notation

\[ Y = X\beta + Z_a a + Z_b b + \epsilon \]

where

\[ Y = \begin{pmatrix} Y_{11} \\ Y_{12} \\ Y_{13} \\ \vdots \\ Y_{53} \end{pmatrix}. \]

Give the matrices \( X, Z_a \) and \( Z_b \), and the vectors \( a, b \) and \( \epsilon \).

(7 marks)

(ii) The data are in the following format:

```r
> head(data)
```

```
Weight  Days  RatID
1 9.56   1 1
2 19.70  7 1
3 35.80 14 1
4 12.45  1 2
5 25.85  7 2
```

What R command would you use to fit the model to the data?

(2 marks)

End of Question Paper
Tables of Percentage Points (also known as Quantiles or Critical Values) for Three Standard Distributions

The tables contain values of quantiles $q$ such that $P[X \leq q] = p$ for various probabilities $p$ when $X$ has the specified distribution (which may depend on particular degrees of freedom $\nu$). In these tables, $p$ has been expressed as a percentage rather than a decimal. The relevant R commands for generating the $q$ are also shown. For the $N(0,1)$ distribution, the tabulated function is also known as the $\Phi^{-1}$ function.

### STANDARD NORMAL DISTRIBUTION PERCENTAGE POINTS

$qnorm(p)$ where $p$ is percentage, e.g. for 95%, $p = 0.95$

<table>
<thead>
<tr>
<th>$p$</th>
<th>60.0%</th>
<th>66.7%</th>
<th>75.0%</th>
<th>80.0%</th>
<th>87.5%</th>
<th>90.0%</th>
<th>95.0%</th>
<th>97.5%</th>
<th>99.0%</th>
<th>99.5%</th>
<th>99.9%</th>
</tr>
</thead>
<tbody>
<tr>
<td>$qnorm$</td>
<td>0.253</td>
<td>0.431</td>
<td>0.674</td>
<td>0.842</td>
<td>1.150</td>
<td>1.282</td>
<td>1.645</td>
<td>1.960</td>
<td>2.326</td>
<td>2.576</td>
<td>3.090</td>
</tr>
</tbody>
</table>

### CHI-SQUARED PERCENTAGE POINTS

$qchisq(p, \nu)$ where $p$ is percentage, e.g. for 95%, $p = 0.95$

<table>
<thead>
<tr>
<th>$\nu$</th>
<th>60.0%</th>
<th>66.7%</th>
<th>75.0%</th>
<th>80.0%</th>
<th>87.5%</th>
<th>90.0%</th>
<th>95.0%</th>
<th>97.5%</th>
<th>99.0%</th>
<th>99.5%</th>
<th>99.9%</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.708</td>
<td>0.936</td>
<td>1.323</td>
<td>1.642</td>
<td>2.354</td>
<td>2.706</td>
<td>3.841</td>
<td>5.024</td>
<td>6.365</td>
<td>7.879</td>
<td>10.828</td>
</tr>
</tbody>
</table>
### Student's t Percentage Points

qt(p, ν) where p is percentage, e.g. for 95%, p = 0.95

<table>
<thead>
<tr>
<th>ν</th>
<th>60.0%</th>
<th>66.7%</th>
<th>75.0%</th>
<th>80.0%</th>
<th>87.5%</th>
<th>90.0%</th>
<th>95.0%</th>
<th>97.5%</th>
<th>99.0%</th>
<th>99.5%</th>
<th>99.9%</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.325</td>
<td>0.577</td>
<td>1.000</td>
<td>1.376</td>
<td>2.414</td>
<td>3.078</td>
<td>6.314</td>
<td>12.706</td>
<td>31.821</td>
<td>63.657</td>
<td>318.31</td>
</tr>
<tr>
<td>2</td>
<td>0.289</td>
<td>0.500</td>
<td>0.816</td>
<td>1.601</td>
<td>1.866</td>
<td>2.920</td>
<td>4.303</td>
<td>6.965</td>
<td>9.925</td>
<td>22.327</td>
<td></td>
</tr>
<tr>
<td>3</td>
<td>0.277</td>
<td>0.476</td>
<td>0.765</td>
<td>1.423</td>
<td>1.638</td>
<td>2.353</td>
<td>3.182</td>
<td>4.541</td>
<td>5.841</td>
<td>10.215</td>
<td></td>
</tr>
<tr>
<td>4</td>
<td>0.271</td>
<td>0.464</td>
<td>0.741</td>
<td>1.344</td>
<td>1.533</td>
<td>2.132</td>
<td>2.776</td>
<td>3.747</td>
<td>4.604</td>
<td>7.173</td>
<td></td>
</tr>
<tr>
<td>5</td>
<td>0.267</td>
<td>0.457</td>
<td>0.727</td>
<td>1.291</td>
<td>1.400</td>
<td>2.015</td>
<td>2.571</td>
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