Data provided: Tables of critical values

MAS6003

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Spring Semester 2018–2019

Linear Models — 3 hours

Marks will be awarded for your best ***five*** *answers.*

RESTRICTED OPEN BOOK EXAMINATION
Candidates may bring to the examination lecture notes and associated lecture material (but no textbooks) plus a calculator that conforms to University regulations.
There are 100 marks available on the paper.

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student

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1 A data set, `weight.data` in `R`, on 30 chicks of a particular species of bird at age 2 months gives `weight`, the weight of the chick in grams, `temp`, the average temperature in Celsius over the 2 month period, and `rain`, the total rainfall in millimetres over the 2 month period.

(a) A linear model was fitted to the data in `R` using the command

```
weight.lm <- lm(weight~temp+rain,data=weight.data)
```

giving output

```
Coefficients:
(Intercept)         temp         rain
  -1120.782      157.857       -4.698
```

The first observation had average temperature 11.7°C and rainfall 15.5mm, and the observed weight of the chick was 435g. Calculate the fitted value and the residual for this observation. ***(2 marks)***

(b) Following on from (a), the command

```
anova(weight.lm)
```

was entered into `R`, producing the following output:

```
Analysis of Variance Table

Response: weight
          Df  Sum Sq Mean Sq F value    Pr(>F)
temp       1 4788381 4788381  19.300 0.0001553 ***
rain       1 1003809 1003809   4.046 0.0543566 .
Residuals 27 6698742  248102
```

Based on this output,

(i) assess the evidence for the inclusion of temperature in the model against the null hypothesis where neither temperature nor rainfall is included; ***(2 marks)***

(ii) assess the evidence for the inclusion of rainfall in the model given that temperature is already included. ***(2 marks)***

(c) Following on from (a) and (b), the code

```
par(mfrow=c(2,2))
plot(weight.lm$resid,xlab="index",ylab="residuals",main="Index plot")
qqnorm(weight.lm$resid,main="QQ - plot")
hist(weight.lm$resid,xlab="Residuals",main="Histogram")
plot(weight.lm$fit,weight.lm$resid,xlab="Fitted values",
 ylab="Residuals",main="Residuals versus fitted values")
```

was entered into `R`, producing the plots in Figure 1. Using the residual plots, comment on whether there appear to be any problems with the assumptions made for a linear model. ***(4 marks)***

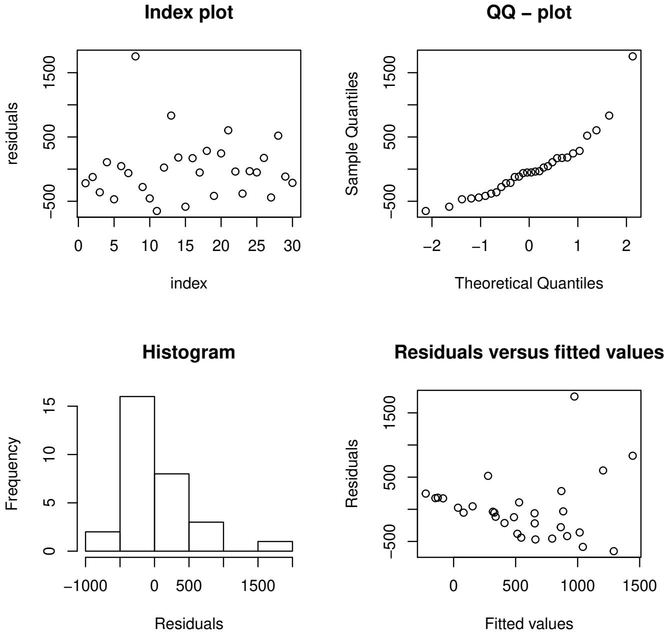

Figure 1: Residual plots for the bird weight data.

1 (continued)

(d) A new model was fitted in R using the command

```
weight.log.lm <- lm(log(weight)~temp+rain,data=weight.data)
```

(i) The output from the new model is

```
Call:
lm(formula = log(weight) ~ temp + rain, data = weight.data)

Coefficients:
(Intercept)         temp         rain
    1.81651      0.36827     -0.01282
```

For the first observation of the data given in (a), calculate the fitted value for the natural log of the weight in the new model and the corresponding residual. ***(3 marks)***

(ii) Residual plots for the new model are shown in Figure 2. Compare these plots with the ones for the original model. Does this model appear to be an improvement? You should give a reason for your answer. ***(4 marks)***

 Question 1 continued on next page

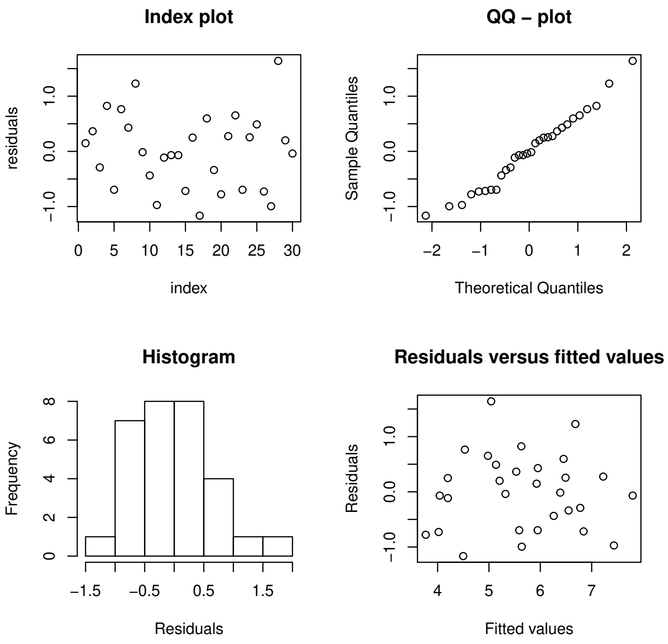

Figure 2: Residual plots for the transformed bird weight data.

1 (continued)

(iii) The code

```
anova(weight.log.lm)
```

gave the output

```
Analysis of Variance Table

Response: log(weight)
          Df  Sum Sq Mean Sq F value    Pr(>F)
temp       1 26.8556 26.8556  54.189 6.427e-08 ***
rain       1  7.4756  7.4756  15.084 0.0006013 ***
Residuals 27 13.3809  0.4956
```

Comparing this to the output from (b), how would your answers for (b)(i) and (b)(ii) change with this output? ***(3 marks)***

2 A model for data $\boldsymbol{y}$ is that each observation y_i is an observation from a Negative Binomial distribution with parameters 2 and p_i so that its probability mass function is

$$f(y_i; p_i) = (y_i - 1)p_i^2(1 - p_i)^{y_i - 2}.$$

(a) Show that the probability mass function can be put into the standard form for a Generalized Linear Model,

$$f(y_i; \theta_i, \phi) = \exp\left\{w_i \frac{y_i\theta_i - b(\theta_i)}{\phi} + c(y_i, \phi)\right\},$$

with $w_i = \phi = 1$, and identify the functions b and c. You should explain the relationship between the quantities θ_i and p_i. ***(8 marks)***

(b) (i) Use the function b to show that the mean of y_i is $\dfrac{2}{1 - e^{\theta_i}}$. ***(2 marks)***

(ii) Give the variance of the distribution of the y_i in terms of θ_i. ***(2 marks)***

(iii) Hence also give the mean and variance of y_i in terms of p_i. ***(2 marks)***

(c) What would the canonical link function be for this Generalized Linear Model? ***(3 marks)***

(d) Suppose that the linear predictor is of the form $\eta_i = \beta_0 + \beta_1 x_i + \beta_2 z_i$, for known vectors of explanatory variables $\boldsymbol{x}$ and $\boldsymbol{z}$. Assuming that the canonical link is used, find the estimated value of p_i for an observation with $x_i = 1$ and $z_i = 2$ for a model with $\beta_0 = -8$ and $\beta_1 = \beta_2 = 1$. ***(3 marks)***

3 An agricultural experiment was carried out to investigate the success of a type of crop, testing the effects of amount of fertilizer and whether the seeds were fresh or not. For each combination of amount of fertilizer and seed freshness, an area of soil was prepared with that amount of fertilizer applied per square metre, 20 seeds were sown, and the number which produced fruit was recorded.

The data are in the following table, and were stored in `R` as `crop.data` with `fert` being the amount of fertilizer per square metre, `storage` indicating whether the seeds had been stored for a year (coded as `1`) or not (coded as `0`), `prop.fruit` indicating the proportion out of the 20 which produced fruit, and `n` being 20 for all observations.

	Amount of fertilizer applied (g m^{-2})					
	0	20	40	60	80	100
Fresh	12	9	16	17	18	20
Stored for one year	4	8	10	13	15	17

Two Generalized Linear Models with Binomial family and logit link were fitted using the following code.

```
model.fert <- glm(prop.fruit~fert,family=binomial,weights=n,data=crop.data)
model.both <- glm(prop.fruit~fert+factor(storage),
 family=binomial,weights=n,data=crop.data)
```

Some R commands and some of the output they produced follow.

```
> summary(model.fert)

Coefficients:
             Estimate Std. Error z value Pr(>|z|)
(Intercept) -0.602470   0.245178  -2.457    0.014 *
fert         0.028487   0.004847   5.878 4.16e-09 ***

    Null deviance: 62.558  on 11  degrees of freedom
Residual deviance: 20.693  on 10  degrees of freedom
AIC: 59.331
```

```
> summary(model.both)

Coefficients:
                 Estimate Std. Error z value Pr(>|z|)
(Intercept)      -0.06123    0.28869  -0.212 0.832029
fert              0.03046    0.00510   5.972 2.34e-09 ***
factor(storage)1 -1.16689    0.31888  -3.659 0.000253 ***

    Null deviance: 62.5585  on 11  degrees of freedom
Residual deviance:  6.4157  on  9  degrees of freedom
AIC: 47.054
```

(a) Using the output above, assess the evidence for

 Question 3 continued on next page

3 (continued)

(i) whether the model with both storage and fertilizer is an improvement on the null model; ***(3 marks)***

(ii) whether storage is needed in the model if the model already includes amount of fertilizer. ***(2 marks)***

(b) In the fitted model with both storage and fertilizer:

(i) Give the odds ratio for success for seeds which have been stored for a year compared with those which are fresh. ***(3 marks)***

(ii) Give an approximate 95% confidence interval for the odds ratio you calculated in (b)(i). ***(2 marks)***

(c) In the fitted model with both storage and fertilizer:

(i) What is the estimated probability that a seed will be successful if it is fresh and 50g per square metre of the fertilizer is used? ***(2 marks)***

(ii) What is the estimated probability that a seed will be successful if it has been stored for a year and 200g per square metre of the fertilizer is used? ***(2 marks)***

(iii) Do you think your answers to (c)(i) and (c)(ii) would give good predictions if these experiments were actually carried out? Give reasons for your answers. ***(4 marks)***

(d) If the model given by `model.both` had given a poor fit, what suggestions might you have to find a better model for these data? Give two suggestions. ***(2 marks)***

4 A survey is carried out to determine the vitamin C concentration in a specific variety of apple. Ten apple trees are selected at random from an orchard, and then three apples are selected from each tree. Two sample pieces of each apple are then selected at random, and the vitamin C concentration in each piece is then measured in a laboratory. The structure of the dataset is as follows:

```
>
> str(appledata)
'data.frame': 60 obs. of  3 variables:
 $ vitC : num  1.203 1 0.35 0.371 -0.296 ...
 $ tree : Factor w/ 10 levels "1","2","3","4",..: 1 1 1 1 1 1 2 2 2 2 ...
 $ apple: Factor w/ 3 levels "1","2","3": 1 1 2 2 3 3 1 1 2 2 ...
>
> head(appledata, 10)
       vitC tree apple
1  27.36133    1     1
2  27.23714    1     1
3  26.47704    1     2
4  26.49594    1     2
5  26.52272    1     3
6  26.57639    1     3
7  26.08871    2     1
8  26.00395    2     1
9  27.63607    2     2
10 27.78275    2     2
>
```

The following R command was used to fit a mixed effect model (model 1) to the data.

```
>
> library(lme4)
> model1 <- lmer(vitC ~ 1 + (1| tree/apple), data=appledata)
>
```

(a) (i) Let V_{ijk} be the vitamin C concentration measured in piece k for apple j for tree i. Write down the algebraic specification of the model that has been fitted to the data making sure you give the distribution of any random effects.

(4 marks)

 Question 4 continued on next page

4 (continued)

(ii) The summary of the fitted model (model 1) is given below. Use the R output to give the estimated values for each parameter in your answer to part (i).

```
>
> summary(model1)
Linear mixed model fit by REML ['lmerMod']
Formula: vitC ~ 1 + (1 | tree/apple)
   Data: appledata

REML criterion at convergence: 103

Scaled residuals:
     Min       1Q   Median       3Q      Max
-1.80846 -0.53225 -0.03586  0.48503  1.87329

Random effects:
 Groups     Name        Variance  Std.Dev.
 apple:tree (Intercept) 1.098e+00  1.04774
 tree       (Intercept) 1.417e+02 11.90539
 Residual               7.933e-03  0.08907
Number of obs: 60, groups:  apple:tree, 30; tree, 10

Fixed effects:
            Estimate Std. Error t value
(Intercept)    18.80   REMOVED   4.987
>
```

(2 marks)

(b) A simpler model (model 2) is fitted using the command

```
>
> model2 <- lmer(vitC~ 1 + (1| tree), data=appledata, REML=F)
>
```

Using the R output below, conduct a hypothesis test to choose between model 1 and model 2. Justify your approach.

```
> model1 <- lmer(vitC~ 1 + (1| tree/apple), data=appledata, REML=F)
> logLik(model1)
'log Lik.' -53.70089 (df=4)
> logLik(model2)
'log Lik.' -115.3253 (df=3)
```

(4 marks)

4 (continued)

(c) Interest lies in the average vitamin C content for the population of apples from this particular type of tree. The standard error of this estimate has been removed from the R output above. Compute the standard error (for model 1) using the variance estimates provided.

Hint: The estimate of the fixed effect is given by the sample mean of the data.

(7 marks)

(d) Briefly describe what diagnostic checks you could use to validate this mixed effect model.

(3 marks)

5 (a) The University are interested in whether the number of emails received by different lecturers varies between the different faculties. They collect data from n lecturers on the number of emails received each day. Let $\mathbf{x} = (x_1, \ldots, x_n)^\top$ be the recorded number of emails received by each lecturer.

A simple statistical model would be that the number of emails received has a Poisson distribution. However, there is reason to believe that lecturers in Arts-based subjects receive emails at a different rate to lecturers in Science-based subjects, and so we will assume that

$$x_i \sim \begin{cases} Poisson(\lambda) & \text{for Arts subject lecturers} \\ Poisson(\mu) & \text{for Science lecturers.} \end{cases}$$

Unfortunately, the information about which faculty each lecturer worked in was lost. Let $0 \leq w \leq 1$ be the unknown proportion of Science-based lecturers in the data set.

We may thus assume that the x_i are samples from the following mixture distribution:

$$x_i \sim \begin{cases} Poisson(\lambda) & \text{with probability } 1 - w, \\ Poisson(\mu) & \text{with probability } w. \end{cases}$$

The corresponding 'missing' variables $\mathbf{Y} = \{Y_1, \ldots, Y_n\}$ are defined as follows:

$$Y_i = \begin{cases} 0 & \text{if lecturer i is in the Arts faculty,} \\ 1 & \text{if lecturer i is in the Science faculty.} \end{cases}$$

Define $\theta = (w, \mu, \lambda)$.

(i) Show that the log-likelihood of θ given the complete data $(\mathbf{x}, \mathbf{Y})$ is

$$l(\theta; \mathbf{x}, \mathbf{Y}) = -\mu \sum Y_i - \lambda \sum (1 - Y_i) + \log \lambda \sum (1 - Y_i) x_i + \log \mu \sum Y_i x_i \\ - \sum \log x_i! + \log w \sum Y_i + \log(1 - w) \sum (1 - Y_i)$$

where all sums are over $i = 1, \ldots, n$.

Hint: the probability mass function of a Poisson(λ) random variable is

$$\mathbb{P}(X = x) = \frac{e^{-\lambda} \lambda^x}{x!} \quad \text{for } x = 0, 1, 2, \ldots$$

(5 marks)

(ii) Using Bayes' theorem, show that

$$\mathbb{E}(Y_i | x_i, \theta) = \frac{w \mu^{x_i} e^{-\mu}}{w \mu^{x_i} e^{-\mu} + (1 - w) \lambda^{x_i} e^{-\lambda}}.$$

Denote this quantity by p_i.

(4 marks)

 Question 5 continued on next page

5 (continued)

(iii) The EM algorithm is to be used to obtain the maximum likelihood estimator $\hat{\theta} = (\hat{w}, \hat{\mu}, \hat{\lambda})$ of θ, given the data $\mathbf{x}$. Let the estimate of $\hat{\theta}$ after m iterations of the EM algorithm be denoted $\theta^{(m)}$. By maximising

$$Q(\theta|\theta^{(m)}) = \mathbb{E}[l(\theta; \mathbf{x}, \mathbf{Y})|\mathbf{x}, \theta^{(m)}],$$

show that the updated estimates of $\hat{\phi}, \hat{\mu}, \hat{\lambda}$ are

$$\begin{aligned} w^{(m+1)} &= \frac{\sum_{i=1}^n p_i}{n}, \\ \lambda^{(m+1)} &= \frac{\sum_{i=1}^n x_i(1-p_i)}{\sum_{i=1}^n (1-p_i)}, \\ \mu^{(m+1)} &= \frac{\sum_{i=1}^n x_i p_i}{\sum_{i=1}^n p_i}, \end{aligned}$$

where $p_i = \mathbb{E}(Y_i|x_i, \theta)$ is your expression derived in part (ii).

(5 marks)

(iv) Give an intuitive explanation of the three updates in part (iii).

(3 marks)

(b) A survey of voting intentions in conducted. Each person is asked for the political party they intend to vote for in the next general election, the city they live in, and their yearly income. Some values are missing for each covariate, but for different reasons in each case.

In each of the following cases, say whether the data are missing completely at random (MCAR), missing at random (MAR), or not missing at random (NMAR):

(i) Some respondents refuse to say who they will vote for. The person conducting the interview suspects that people who intend to vote for far right parties are less likely to provide this information than the general population.

(ii) Due to a corrupted file, the home city of every fourth person in the survey is lost.

(iii) Some respondents refuse to reveal their income. It is known that older people are less likely than the general population to reveal their income, regardless of what their income is.

(3 marks)

6 (a) Consider a dataset on the record winning times of 35 hill races in Scotland. The covariates are

- `dist` - distance in miles
- `climb` - total height gained during the route, in feet.
- `time` - record time in minutes

The output below shows the structure of the dataset.

```
> str(hill)
'data.frame': 35 obs. of  3 variables:
 $ dist : num  NA 6 6 7.5 8 8 16 NA 5 NA ...
 $ climb: int  650 2500 NA 800 3070 NA 7500 800 800 650 ...
 $ time : num  16.1 NA 33.6 45.6 62.3 ...
>
> head(hill)
             dist climb   time
Greenmantle    NA   650 16.083
Carnethy      6.0  2500     NA
Craig Dunain  6.0    NA 33.650
Ben Rha       7.5   800 45.600
Ben Lomond    8.0  3070 62.267
Goatfell      8.0    NA 73.217
```

(i) The following R command is used.

```
> hill.mice <- mice(hill, m=5, method=c('norm', 'norm', 'mean'))
```

Describe in detail the statistical procedure that is used to fill in the missing values.

(6 marks)

6 (continued)

(ii) Interest lies in the coefficient of `dist` (β_1) in the linear regression model

$$\texttt{time} = \beta_0 + \beta_1 \texttt{dist} + \beta_2 \texttt{climb} + \epsilon.$$

Use the R output below to calculate an expected value of β_1 and its standard error.

```
> fit.mice <- with(hill.mice, lm(time ~ dist+climb))
>
> (coefs = sapply(fit.mice$analyses, coef))
              [,1]  [,2]  [,3]  [,4]  [,5]
(Intercept) -3.12 1.27 -1.14 -3.47 -2.29
dist          6.46 6.15  6.50  6.16  6.41
climb         0.01 0.01  0.01  0.01  0.01
>
> apply(coefs, 1, var)
(Intercept)        dist       climb
    3.7e+00     2.9e-02     6.9e-08
>
> sapply(fit.mice$analyses, function(x) vcov(x)[2,2])
[1] 0.49 0.56 0.47 0.52 0.37
>
```

(5 marks)

(b) A longitudinal study is conducted on the weight gain of rats in the first two weeks of life. Let y_{ij} be the weight of the i^{th} rat at the j^{th} measurement where $i = 1, \ldots, 5$, and $j = 1, \ldots, 3$. Let x_j be the age of the rat in days at the j^{th} measurement with $x_1 = 1, x_2 = 7, x_3 = 14$. We will use the model

$$y_{ij} = \alpha + a_i + (\beta + b_i)x_j + \epsilon_{ij}$$

where $a_i \sim N(0, \sigma_a^2)$, $b_i \sim N(0, \sigma_b^2)$ and $\epsilon_{ij} \sim N(0, \sigma^2)$, with all random effects independent of each other.

 Question 6 continued on next page

6 (continued)

(i) Write the model in matrix notation

$$\mathbf{Y} = X\boldsymbol{\beta} + Z_a\mathbf{a} + Z_b\mathbf{b} + \boldsymbol{\epsilon}$$

where

$$\mathbf{Y} = \begin{pmatrix} Y_{11} \\ Y_{12} \\ Y_{13} \\ Y_{21} \\ \vdots \\ Y_{53} \end{pmatrix}.$$

Give the matrices X, Z_a and Z_b, and the vectors $\mathbf{a}$, $\mathbf{b}$ and $\boldsymbol{\epsilon}$.

(7 marks)

(ii) The data are in the following format:

```
> head(data)
    Weight  Days RatID
1     9.56    1    1
2    19.70    7    1
3    35.80    14   1
4    12.45    1    2
5    25.85    7    2
>
```

What R command would you use to fit the model to the data?

(2 marks)

End of Question Paper

Tables of Percentage Points (also known as Quantiles or Critical Values) for Three Standard Distributions

The tables contain values of quantiles q such that $P[X \leq q] = p$ for various probabilities p when X has the specified distribution (which may depend on particular degrees of freedom ν). In these tables, p has been expressed as a percentage rather than a decimal. The relevant R commands for generating the q are also shown. For the $N(0,1)$ distribution, the tabulated function is also known as the Φ^{-1} function.

STANDARD NORMAL DISTRIBUTION PERCENTAGE POINTS

`qnorm`(p) where p is percentage, e.g. for 95%, $p = 0.95$

	60.0%	66.7%	75.0%	80.0%	87.5%	90.0%	95.0%	97.5%	99.0%	99.5%	99.9%
qnorm	0.253	0.431	0.674	0.842	1.150	1.282	1.645	1.960	2.326	2.576	3.090

CHI-SQUARED PERCENTAGE POINTS

`qchisq`(p, ν) where p is percentage, e.g. for 95%, $p = 0.95$

ν	60.0%	66.7%	75.0%	80.0%	87.5%	90.0%	95.0%	97.5%	99.0%	99.5%	99.9%
1	0.708	0.936	1.323	1.642	2.354	2.706	3.841	5.024	6.635	7.879	10.828
2	1.833	2.197	2.773	3.219	4.159	4.605	5.991	7.378	9.210	10.597	13.816
3	2.946	3.405	4.108	4.642	5.739	6.251	7.815	9.348	11.345	12.838	16.266
4	4.045	4.579	5.385	5.989	7.214	7.779	9.488	11.143	13.277	14.860	18.467
5	5.132	5.730	6.626	7.289	8.625	9.236	11.070	12.833	15.086	16.750	20.515
6	6.211	6.867	7.841	8.558	9.992	10.645	12.592	14.449	16.812	18.548	22.458
7	7.283	7.992	9.037	9.803	11.326	12.017	14.067	16.013	18.475	20.278	24.322
8	8.351	9.107	10.219	11.030	12.636	13.362	15.507	17.535	20.090	21.955	26.125
9	9.414	10.215	11.389	12.242	13.926	14.684	16.919	19.023	21.666	23.589	27.877
10	10.473	11.317	12.549	13.442	15.198	15.987	18.307	20.483	23.209	25.188	29.588

STUDENT'S t PERCENTAGE POINTS

`qt(`p, ν`)` where p is percentage, e.g. for 95%, $p = 0.95$

ν	60.0%	66.7%	75.0%	80.0%	87.5%	90.0%	95.0%	97.5%	99.0%	99.5%	99.9%
1	0.325	0.577	1.000	1.376	2.414	3.078	6.314	12.706	31.821	63.657	318.31
2	0.289	0.500	0.816	1.061	1.604	1.886	2.920	4.303	6.965	9.925	22.327
3	0.277	0.476	0.765	0.978	1.423	1.638	2.353	3.182	4.541	5.841	10.215
4	0.271	0.464	0.741	0.941	1.344	1.533	2.132	2.776	3.747	4.604	7.173
5	0.267	0.457	0.727	0.920	1.301	1.476	2.015	2.571	3.365	4.032	5.893
6	0.265	0.453	0.718	0.906	1.273	1.440	1.943	2.447	3.143	3.707	5.208
7	0.263	0.449	0.711	0.896	1.254	1.415	1.895	2.365	2.998	3.499	4.785
8	0.262	0.447	0.706	0.889	1.240	1.397	1.860	2.306	2.896	3.355	4.501
9	0.261	0.445	0.703	0.883	1.230	1.383	1.833	2.262	2.821	3.250	4.297
10	0.260	0.444	0.700	0.879	1.221	1.372	1.812	2.228	2.764	3.169	4.144
11	0.260	0.443	0.697	0.876	1.214	1.363	1.796	2.201	2.718	3.106	4.025
12	0.259	0.442	0.695	0.873	1.209	1.356	1.782	2.179	2.681	3.055	3.930
13	0.259	0.441	0.694	0.870	1.204	1.350	1.771	2.160	2.650	3.012	3.852
14	0.258	0.440	0.692	0.868	1.200	1.345	1.761	2.145	2.624	2.977	3.787
15	0.258	0.439	0.691	0.866	1.197	1.341	1.753	2.131	2.602	2.947	3.733
16	0.258	0.439	0.690	0.865	1.194	1.337	1.746	2.120	2.583	2.921	3.686
17	0.257	0.438	0.689	0.863	1.191	1.333	1.740	2.110	2.567	2.898	3.646
18	0.257	0.438	0.688	0.862	1.189	1.330	1.734	2.101	2.552	2.878	3.610
19	0.257	0.438	0.688	0.861	1.187	1.328	1.729	2.093	2.539	2.861	3.579
20	0.257	0.437	0.687	0.860	1.185	1.325	1.725	2.086	2.528	2.845	3.552
21	0.257	0.437	0.686	0.859	1.183	1.323	1.721	2.080	2.518	2.831	3.527
22	0.256	0.437	0.686	0.858	1.182	1.321	1.717	2.074	2.508	2.819	3.505
23	0.256	0.436	0.685	0.858	1.180	1.319	1.714	2.069	2.500	2.807	3.485
24	0.256	0.436	0.685	0.857	1.179	1.318	1.711	2.064	2.492	2.797	3.467
25	0.256	0.436	0.684	0.856	1.178	1.316	1.708	2.060	2.485	2.787	3.450
26	0.256	0.436	0.684	0.856	1.177	1.315	1.706	2.056	2.479	2.779	3.435
27	0.256	0.435	0.684	0.855	1.176	1.314	1.703	2.052	2.473	2.771	3.421
28	0.256	0.435	0.683	0.855	1.175	1.313	1.701	2.048	2.467	2.763	3.408
29	0.256	0.435	0.683	0.854	1.174	1.311	1.699	2.045	2.462	2.756	3.396
30	0.256	0.435	0.683	0.854	1.173	1.310	1.697	2.042	2.457	2.750	3.385
35	0.255	0.434	0.682	0.852	1.170	1.306	1.690	2.030	2.438	2.724	3.340
40	0.255	0.434	0.681	0.851	1.167	1.303	1.684	2.021	2.423	2.704	3.307
45	0.255	0.434	0.680	0.850	1.165	1.301	1.679	2.014	2.412	2.690	3.281
50	0.255	0.433	0.679	0.849	1.164	1.299	1.676	2.009	2.403	2.678	3.261
55	0.255	0.433	0.679	0.848	1.163	1.297	1.673	2.004	2.396	2.668	3.245
60	0.254	0.433	0.679	0.848	1.162	1.296	1.671	2.000	2.390	2.660	3.232
∞	0.253	0.431	0.674	0.842	1.150	1.282	1.645	1.960	2.326	2.576	3.090