



Answer **four** questions. If you answer more than four questions, only your best four will be counted.

Throughout this paper, unless otherwise stated, all vector spaces are either over the field of real numbers,  $\mathbb{R}$ , or the field of complex numbers,  $\mathbb{C}$ . We write  $\mathbb{K}$  to denote either  $\mathbb{R}$  or  $\mathbb{C}$ .

**1** (i) (a) Let  $C[0, 1]$  be the complex vector space of continuous functions  $f: [0, 1] \rightarrow \mathbb{C}$ . Prove that we have a norm on  $C([0, 1])$  defined by the formula

$$\|f\|_0 = \sup\{|f(t)| \mid t \in [0, 1]\}.$$

(b) Let  $C_0^{(1)}[0, 1]$  be the complex vector space of all differentiable functions  $f: [0, 1] \rightarrow \mathbb{C}$  with continuous derivative, such that  $f(0) = 0$ . Prove that we have a norm on  $C_0^{(1)}[0, 1]$  defined by the formula

$$\|f\|_1 = \|f\|_0 + \|f'\|_0$$

(6 marks)

(ii) What does it mean for a normed vector space to be a *Banach space*. Prove that  $C[0, 1]$  is a Banach space. (9 marks)

(iii) Let  $V$  and  $W$  be normed vector spaces. Say what is meant by a *bounded linear map*  $T: V \rightarrow W$ . Prove that we have bounded linear maps  $D: C_0^{(1)}[0, 1] \rightarrow C[0, 1]$  and  $I: C[0, 1] \rightarrow C_0^{(1)}[0, 1]$  defined by the formulae

$$D(f) = f' \quad I(f)(x) = \int_0^x f(t) dt$$

respectively.

(5 marks)

(iv) Is the space  $C_0^{(1)}[0, 1]$  a Banach space? Justify your answer. (5 marks)

2 (i) State Zorn's lemma, including definition of the terms *maximal* and *upper bound*. (4 marks)

(ii) Let  $V$  be a vector space over the field  $\mathbb{K}$ , and let  $W$  be a subspace. Let  $f: W \rightarrow \mathbb{K}$  be a linear map. Prove that there exists a linear map  $F: V \rightarrow \mathbb{K}$  such that  $F(w) = f(w)$  for all  $w \in W$ . (7 marks)

(iii) Let  $V$  be a normed vector space over the field  $\mathbb{K}$ . Define the dual space  $V^*$  and the norm of a bounded linear map  $f: V \rightarrow \mathbb{K}$ . Prove that the dual space  $V^*$  is a normed vector space under this norm. (5 marks)

(iv) State the Hahn-Banach theorem. (3 marks)

(v) Prove that the linear map  $\tau: V \rightarrow (V^*)^*$  defined by the formula

$$\tau(v)(f) = f(v) \quad f \in V^*, v \in V$$

is an isometry. (6 marks)

3 (i) Let  $V$  and  $W$  be Banach spaces. State what is meant by a linear map  $T: V \rightarrow W$  being *open* and having *closed graph*. State the open mapping and closed graph theorems, and use the open mapping theorem to prove the closed graph theorem. (9 marks)

(ii) Let  $H$  be a Hilbert space. Let  $T: H \rightarrow H$  be a linear map. Suppose that  $\langle Tu, v \rangle = \langle u, Tv \rangle$  for all  $u, v \in H$ . Use the closed graph theorem to show that  $T$  is continuous. (8 marks)

(iii) Let  $H$  be a Hilbert space, and let  $T: H \rightarrow H$  be a bounded linear map. Define the *adjoint*  $T^*: H \rightarrow H$ . Compute the adjoint of the maps  $S, T: L^2(\mathbb{R}) \rightarrow L^2(\mathbb{R})$  defined by the formulae

$$S(f)(x) = f(x + 1) \quad T(g)(x) = e^{ix}g(x - 1)$$

respectively, where  $x \in \mathbb{R}$ . (8 marks)

4 (i) Let  $A$  be a complex unital Banach algebra, and let  $x \in A$ . Define the *spectrum* of  $x$ . (2 marks)

(ii) Let  $x \in A$  satisfy the inequality  $\|x\| < 1$ . Prove that the element  $1 - x$  is invertible. Deduce that if  $\lambda \in \text{Spectrum}(x)$  then  $|\lambda| < \|x\|$ . (10 marks)

(iii) Let  $H$  be a Hilbert space, and let  $T: H \rightarrow H$  be a bounded linear map. Prove that if  $\lambda$  is an eigenvalue of  $T$ , then  $\lambda \in \text{Spectrum}(T)$ . (3 marks)

(iv) Let  $R: \ell^2 \rightarrow \ell^2$  be the right shift operator. Prove that  $R$  has no eigenvalues. Find the spectrum of  $R$ . (10 marks)

5 (i) Define what is meant by the statement that a linear map between normed vector spaces is a *compact operator*. (2 marks)

(ii) Let  $K: V \rightarrow W$  be a compact operator between normed vector spaces  $V$  and  $W$ . Let  $(x_n)$  be a bounded sequence in  $V$ . Prove that  $(Kx_n)$  has a convergent subsequence. (4 marks)

(iii) Prove that any bounded linear map with finite-dimensional image is compact. You may if you wish use the Heine-Borel theorem without proof. (4 marks)

(iv) Prove that the operator  $S: \ell^2 \rightarrow \ell^2$  defined by the formula

$$S(a_1, a_2, a_3, \dots) = (a_1, \frac{a_2}{2^2}, \frac{a_3}{3^2}, \dots)$$

is compact. You may use without proof here the fact that a norm-limit of a sequence of compact operators is again compact. (5 marks)

(v) What is the definition of a Fredholm operator? (2 marks)

(vi) Define  $T: \ell^2 \rightarrow \ell^2$  by the formula

$$T(a_1, a_2, a_3, \dots) = (a_1 + a_3, \frac{a_2}{2^2} + a_4, \frac{a_3}{3^2} + a_5, \dots)$$

Show that  $T$  is Fredholm, and calculate  $\text{Index}(T)$ . You may use without proof any standard results from the theory of Fredholm operators. (8 marks)

**End of Question Paper**