

SCHOOL OF MATHEMATICS AND STATISTICS

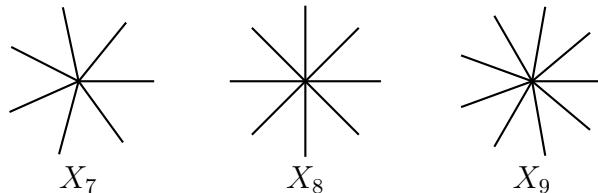
Spring Semester
2018–2019

Algebraic Topology

2 hours 30 minutes

Answer **four** questions. You are advised **not** to answer more than four questions: if you do, only your best four will be counted.

- 1 (a) Given a topological space X , define the set $\pi_0(X)$. You should include a proof that the relevant equivalence relation is in fact an equivalence relation. (8 marks)
- (b) Consider $[0, 1]$ as a based space with 0 as the basepoint. For $n \geq 3$ we define $X_n = \{z \in \mathbb{C} \mid z^n \in [0, 1]\}$:



- (i) For which n and m (with $n, m \geq 3$) is X_n homotopy equivalent to X_m ? (3 marks)
- (ii) For which n and m (with $n, m \geq 3$) is X_n homeomorphic to X_m ? (4 marks)

Justify your answers carefully.

- (c) Give examples as follows, with justification:
- (1) A based space W with $|\pi_1(W)| = 8$. (3 marks)
- (2) A space X with two points $a, b \in X$ such that $\pi_1(X, a)$ is not isomorphic to $\pi_1(X, b)$. (3 marks)
- (3) A space Y such that $H_0(Y) \simeq H_2(Y) \simeq H_4(Y) \simeq H_6(Y) \simeq \mathbb{Z}$ and all other homology groups are trivial. (4 marks)

2 Are the following true or false? Justify your answers.

- (a) S^5 is a Hausdorff space. (4 marks)
- (b) The Klein bottle is a retract of $S^1 \times S^1 \times S^1$. (4 marks)
- (c) There is a connected space X with $\pi_1(X) \simeq \mathbb{Z}/2$ and $H_1(X) \simeq \mathbb{Z}$. (4 marks)
- (d) There is a short exact sequence $\mathbb{Z}/9 \rightarrow \mathbb{Z}/99 \rightarrow \mathbb{Z}/11$. (4 marks)
- (e) If K is a simplicial complex and L is a subcomplex and $H_3(K) = 0$ then $H_3(L) = 0$. (4 marks)
- (f) If K and L are simplicial complexes and $f: |K| \rightarrow |L|$ is a continuous map then there is a simplicial map $s: K \rightarrow L$ such that f is homotopic to $|s|$. (5 marks)

3 Let K and L be abstract simplicial complexes.

- (a) Define what is meant by a *simplicial map* from K to L . (3 marks)
- (b) Let $s, t: K \rightarrow L$ be simplicial maps. Define what it means for s and t to be *directly contiguous*. (3 marks)
- (c) Prove that if s and t are directly contiguous, then the resulting maps $|s|, |t|: |K| \rightarrow |L|$ are homotopic. (3 marks)
- (d) Prove that if s and t are directly contiguous, then the resulting maps $s_*, t_*: H_*(K) \rightarrow H_*(L)$ are the same. (You can prove the main formula just for $n = 3$ rather than general n .) (9 marks)
- (e) How many injective simplicial maps are there from $\partial\Delta^2$ to itself? Show that no two of them are directly contiguous. (7 marks)

4 Let $U_* \xrightarrow{i} V_* \xrightarrow{p} W_*$ be a short exact sequence of chain complexes and chain maps.

- (a) Define what is meant by saying that the above sequence is short exact. (3 marks)

Now recall that a *snake* for the above sequence is a system (c, w, v, u, a) such that

- $c \in H_n(W)$;
- $w \in Z_n(W)$ is a cycle such that $c = [w]$;
- $v \in V_n$ is an element with $p(v) = w$;
- $u \in Z_{n-1}(U)$ is a cycle with $i(u) = d(v) \in V_{n-1}$;
- $a = [u] \in H_{n-1}(U)$.

- (b) Prove that for each $c \in H_n(W)$ there is a snake starting with c . (8 marks)
- (c) Prove that if two snakes have the same starting point, then they also have the same endpoint. (10 marks)
- (d) Suppose that the differential $d: V_{n+1} \rightarrow V_n$ is surjective. Show that any snake starting in $H_n(W)$ ends with zero. (4 marks)

5 Consider a simplicial complex K with subcomplexes L and M such that $K = L \cup M$. Use the following notation for the inclusion maps:

$$\begin{array}{ccc} L \cap M & \xrightarrow{i} & L \\ j \downarrow & & \downarrow f \\ M & \xrightarrow{g} & K. \end{array}$$

- (a) State the Seifert-van Kampen Theorem (in a form applicable to simplicial complexes and subcomplexes as above). (4 marks)
- (b) State the Mayer-Vietoris Theorem. (5 marks)
- (c) State a theorem about the relationship between π_1 and H_1 . (3 marks)
- (d) Suppose that $|L|$, $|M|$ and $|L \cap M|$ are all homotopy equivalent to S^1 . Suppose that the maps i and j both have degree two.
- (1) Find a presentation for $\pi_1|K|$. (3 marks)
- (2) Find $H_*(K)$. In particular, you should express each nonzero group as a direct sum of terms like \mathbb{Z} or \mathbb{Z}/n . (10 marks)

End of Question Paper