The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Semester 2018–2019**

Analytical Dynamics and Classical Field Theory **3 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

1 Define the Lagrangian L of a mechanical system with total kinetic energy T and potential energy V. ***(1 mark)***

Two point particles A and B, each of mass m, are connected by a light inextendible string of length ℓ. The string passes through a hole in a smooth horizontal table. Particle A moves on the table. Particle B moves on a vertical axis through the hole, and lies below the table. Let (r, θ) be plane polar coordinates describing the position of particle A on the table, with the hole at the origin. Assume that the string remains taut during the motion of the particles.

Show that the Lagrangian of the system is

$$L = m\dot{r}^2 + \frac{1}{2}mr^2\dot{\theta}^2 + mg\,(\ell - r)\,,$$

where $\dot{r} = \dfrac{dr}{dt}$ and $\dot{\theta} = \dfrac{d\theta}{dt}$. ***(7 marks)***

Find the specific form of Lagrange's equations that govern the motion of this system. ***(4 marks)***

Hence show that $mr^2\dot{\theta}$ is a constant of the motion and interpret this physically. ***(2 marks)***

Show that the total energy of the system is conserved. ***(6 marks)***

2 (i) Write down Hamilton's equations for a mechanical system with one degree of freedom, having Hamiltonian $H(q, p, t)$, where q is the generalized coordinate, p the conjugate momentum and t is time. ***(2 marks)***

The Poisson bracket of two dynamical variables $A(q, p, t)$ and $B(q, p, t)$ is given by

$$\{A, B\} = \frac{\partial A}{\partial q}\frac{\partial B}{\partial p} - \frac{\partial A}{\partial p}\frac{\partial B}{\partial q}.$$

Show that

$$\frac{dA}{dt} = \frac{\partial A}{\partial t} + \{A, H\}.$$

(3 marks)

Deduce that

$$\frac{dH}{dt} = \frac{\partial H}{\partial t}.$$

(1 mark)

(ii) A mechanical system with one degree of freedom, with generalized coordinate q and generalized velocity $\dot{q}$, has Lagrangian

$$L = \frac{1}{4}\dot{q}^2 + \frac{1}{2q^2}.$$

Find the momentum p conjugate to q and hence the Hamiltonian of the system. ***(3 marks)***

Find Hamilton's equations for the system. ***(2 marks)***

Explain why the Hamiltonian is a constant of the motion. ***(1 mark)***

Use Hamilton's equations to show that the quantity $K = \frac{1}{2}pq - Ht$ is a constant of the motion. ***(3 marks)***

Find $\{K, H\}$ and $\frac{\partial K}{\partial t}$ and verify that

$$\frac{dK}{dt} = \frac{\partial K}{\partial t} + \{K, H\}.$$

(4 marks)

Explain why there are no other independent constants of the motion. ***(1 mark)***

3 Write down the components of the Minkowski metric $\eta_{\mu\nu}$ in an inertial frame. ***(1 mark)***

Two inertial frames have space-time coordinates x^{ν} and $x^{\mu'}$ respectively, where x^{ν} and $x^{\mu'}$ are related by the transformation

$$x^{\mu'} = \Lambda^{\mu}{}_{\nu} x^{\nu},$$

where $\Lambda^{\mu}{}_{\nu}$ are constants.

Write down the condition on the matrix Λ for the above transformation to be a Lorentz transformation. ***(1 mark)***

For the rest of this question you may assume that the Minkowski metric $\eta_{\mu\nu}$ is a tensor of type $(0, 2)$ and that the inverse Minkowski metric $\eta^{\mu\nu}$ is a tensor of type $(2, 0)$.

Let Φ be a scalar field.

Show that the derivative $\partial_{\mu}\Phi$ is a tensor of type $(0, 1)$, that $(\partial_{\mu}\Phi)(\partial^{\mu}\Phi)$ is a Lorentz scalar and that $(\partial_{\mu}\Phi)(\partial_{\nu}\Phi)$ is a tensor of type $(0, 2)$. ***(5 marks)***

The scalar field is governed by the Lagrangian density

$$\mathcal{L}(\Phi, \partial_{\mu}\Phi) = -\frac{1}{2}\left[(\partial_{\mu}\Phi)(\partial^{\mu}\Phi) + V(\Phi)\right]$$

where $V(\Phi)$ is an arbitrary scalar function of Φ.

Find the field equation satisfied by the scalar field Φ. ***(6 marks)***

The stress-energy tensor $T_{\mu\nu}$ of the scalar field is defined by

$$T_{\mu\nu} = 2\frac{\partial \mathcal{L}}{\partial \eta^{\mu\nu}} - \eta_{\mu\nu}\mathcal{L},$$

where, in calculating the partial derivative, $\eta^{\mu\nu}$ is treated as a variable independent of Φ and its derivatives.

Show that

$$T_{\mu\nu} = -(\partial_{\mu}\Phi)(\partial_{\nu}\Phi) + \frac{1}{2}\eta_{\mu\nu}\left[(\partial^{\alpha}\Phi)(\partial_{\alpha}\Phi) + V(\Phi)\right].$$

(1 mark)

Show that $T_{\mu\nu}$ is a tensor of type $(0, 2)$. ***(2 marks)***

Use the scalar field equation for Φ to show that $\partial^{\mu}T_{\mu\nu} = 0$. ***(4 marks)***

4 (i) Let $x^\alpha(\lambda)$ denote a geodesic satisfying $u^\beta \nabla_\beta u^\alpha = 0$, where ∇_β is the metric-compatible covariant derivative, $u^\alpha \equiv \dfrac{dx^\alpha}{d\lambda}$ is the tangent vector, and λ is an affine parameter.

(a) Show that $g_{\mu\nu}u^\mu u^\nu$ is constant along the geodesic. ***(3 marks)***

(b) Let X^α be a Killing vector such that $\nabla_\alpha X_\beta + \nabla_\beta X_\alpha = 0$.
Show that $X_\alpha u^\alpha$ is constant along the geodesic. ***(3 marks)***

(ii) Now consider a Lagrangian L for geodesics on a spherically-symmetric spacetime with coordinates $x^\alpha = [t, r, \theta, \phi]$ given by

$$L = \frac{1}{2} g_{\mu\nu}\dot{x}^\mu \dot{x}^\nu = \frac{1}{2}\left(-f\,\dot{t}^2 + f^{-1}\,\dot{r}^2 + r^2\dot{\theta}^2 + r^2 \sin^2\theta\,\dot{\phi}^2\right),$$

where $f = f(r)$ is a function of r only, and $\dot{t} = \dfrac{dt}{d\lambda}$, $\dot{r} = \dfrac{dr}{d\lambda}$, etc.

(a) Use the Euler-Lagrange equations to show that

$$E \equiv -f\dot{t} \qquad \text{and} \qquad h \equiv r^2 \sin^2\theta\,\dot{\phi}$$

are constants of motion. ***(4 marks)***

(b) For the case of timelike geodesics ($L = -1/2$) in the equatorial plane ($\theta = \pi/2$, $\dot{\theta} = 0$), derive an energy equation in the form

$$\dot{r}^2 = E^2 - V(r), \qquad V(r) \equiv f(r)\left(1 + \frac{h^2}{r^2}\right).$$

(4 marks)

(c) Let $f(r) = (1 - M/r)^2$ for an extremally-charged black hole.
Solve $V'(r) = 0$ to find an expression for h^2 on a *circular orbit* of radius r.
Solve $V''(r) = 0$ and use h^2 to find the radius of the *innermost stable circular orbit* with $r > 2M$. ***(6 marks)***

5 The covariant derivative of a covector v_β is $\nabla_\alpha v_\beta \equiv v_{\beta;\alpha} = v_{\beta,\alpha} - \Gamma^\mu{}_{\beta\alpha} v_\mu$ where $v_{\beta,\alpha} \equiv \partial v_\beta / \partial x^\alpha$. The Christoffel connection is defined by

$$\Gamma^\alpha{}_{\mu\nu} \equiv \frac{1}{2} g^{\alpha\beta} \left(g_{\beta\mu,\nu} + g_{\beta\nu,\mu} - g_{\mu\nu,\beta} \right),$$

and the Riemann tensor is defined by

$$R^\alpha{}_{\beta\gamma\delta} \equiv \Gamma^\alpha{}_{\beta\delta,\gamma} - \Gamma^\alpha{}_{\beta\gamma,\delta} + \Gamma^\alpha{}_{\mu\gamma}\Gamma^\mu{}_{\beta\delta} - \Gamma^\alpha{}_{\mu\delta}\Gamma^\mu{}_{\beta\gamma}.$$

The Ricci tensor is $R_{\mu\nu} \equiv R^\alpha{}_{\mu\alpha\nu}$, and the Ricci scalar is $R \equiv R^\alpha{}_\alpha$.

(i) Use the Christoffel connection and the definition of the covariant derivative to show that

$$\nabla_\alpha g_{\mu\nu} = 0.$$

(3 marks)

(ii) The line element of a spatially-flat time-dependent spacetime is

$$ds^2 = -dt^2 + a^2(t) \left(dx^2 + dy^2 + dz^2 \right),$$

where $a(t)$ is the scale factor.
All of the components of the Christoffel connection are zero, with the exception of $\Gamma^t{}_{xx} = \Gamma^t{}_{yy} = \Gamma^t{}_{zz}$ and $\Gamma^x{}_{tx} = \Gamma^y{}_{ty} = \Gamma^z{}_{tz}$.

(a) Show that $\Gamma^x{}_{tt} = 0$. *(2 marks)*

(b) Show that $\Gamma^t{}_{xx} = a\dot{a}$ where $\dot{a} = da/dt$.
Derive an expression for $\Gamma^x{}_{tx}$. *(4 marks)*

(c) Derive an expression for $R^x{}_{txt}$ in terms of the scale factor.
Show that $R_{tt} = -3\ddot{a}/a$.
Derive an expression for R_{xx}. *(7 marks)*

(d) An inflating universe has a scale factor of $a(t) = e^{\beta t}$, where β is a positive constant.
Using the results of part (ii)(c), and by choosing the constant Λ suitably, show that $R_{\mu\nu} = \Lambda g_{\mu\nu}$. (You may assume without proof that $R_{\mu\nu}$ is diagonal).
Hence write down the Ricci scalar R. *(4 marks)*

End of Question Paper