



The  
University  
Of  
Sheffield.

MAS6446

SCHOOL OF MATHEMATICS AND STATISTICS

Spring Semester 2018–2019

Mathematical Methods and Modelling of Natural  
Systems

2 hours

*Marks will be awarded for your best FOUR answers. The marks awarded to each question or section of question are shown in italics. The total number of marks for the paper is 100.*

- 1 (a) Define  $\Delta(x; \epsilon)$  for  $\epsilon > 0$  by

$$\Delta(x; \epsilon) = \frac{\epsilon}{\pi(x^2 + \epsilon^2)}.$$

Show that

$$\int_{-\infty}^{\infty} \Delta(x; \epsilon) dx = 1. \quad (4 \text{ marks})$$

By considering  $\lim_{\epsilon \rightarrow 0} \int_{-\infty}^{\infty} f(x) \Delta(x - a; \epsilon) dx$  for an arbitrary well-behaved function  $f(x)$  and a constant  $a$ , deduce that we could define the delta function  $\delta(x)$  by

$$\delta(x) = \lim_{\epsilon \rightarrow 0} \frac{\epsilon}{\pi(x^2 + \epsilon^2)}. \quad (5 \text{ marks})$$

- (b) Define an integral  $I(x; \epsilon)$  for real  $x$  and  $\epsilon > 0$  by

$$I(x; \epsilon) = \int_{-\infty}^{\infty} e^{ikx - \epsilon|k|} dk.$$

Evaluate  $I(x; \epsilon)$  and, using the result of part (a), deduce that

$$\int_{-\infty}^{\infty} e^{ikx} dk = 2\pi\delta(x). \quad (6 \text{ marks})$$

Hence show that

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-ikx} \hat{f}(k) dk,$$

where  $\hat{f}(k)$  is the Fourier transform of  $f(x)$ . (4 marks)

- (c) If  $\hat{f}(k) = e^{-k^2/2}$ , use the result of part (b) to find the corresponding  $f(x)$  for real  $x$ . (6 marks)

$$\left[ \begin{array}{l} \text{You may use} \\ \int_{-\infty}^{\infty} e^{-\frac{1}{2}(k+ix)^2} dk = \int_{-\infty}^{\infty} e^{-\frac{1}{2}k^2} dk \quad \text{for real } x, \\ \text{and} \\ \int_{-\infty}^{\infty} e^{-v^2} dv = \sqrt{\pi}. \end{array} \right]$$

- 2 A semi-infinite rod occupies the positive  $x$ -axis. The temperature,  $T(x, t)$ , in the rod satisfies

$$\frac{\partial T}{\partial t} = \kappa \frac{\partial^2 T}{\partial x^2} \quad \text{for } 0 < x < \infty, \quad 0 < t < \infty, \quad (1)$$

where  $\kappa (> 0)$  is the constant conductivity. Initially the rod has zero temperature everywhere, i.e.  $T(x, 0) = 0$ , and for  $t > 0$  the end of the rod is maintained at a constant temperature  $T_0$ , i.e.  $T(0, t) = T_0$ . For  $t > 0$ ,  $T(x, t) \rightarrow 0$  as  $x \rightarrow \infty$ .

The Laplace transform of  $T(x, t)$  is defined by

$$\tilde{T}(x, s) = \mathcal{L}\{T(x, t)\} = \int_0^\infty e^{-st} T(x, t) dt,$$

and exists for  $\text{Re } s > c$  for some constant  $c > 0$ .

- (a) Show that

$$\mathcal{L}\left\{\frac{\partial T}{\partial t}\right\} = s\tilde{T}(x, s) - T(x, 0)$$

for  $\text{Re } s > c$ .

(4 marks)

- (b) By taking the Laplace transform of (1), show that

$$\tilde{T}(x, s) = A(s)e^{(\frac{x}{\kappa})^{1/2}x} + B(s)e^{-(\frac{x}{\kappa})^{1/2}x}$$

for some functions  $A$  and  $B$ .

(5 marks)

By applying the conditions at  $x = 0$  and as  $x \rightarrow \infty$  for  $\text{Re}(s^{1/2}) > 0$ , find  $A(s)$  and  $B(s)$  and, hence,  $\tilde{T}(x, s)$ .

(6 marks)

- (c) Given that the Laplace transform of

$$\int_{\frac{a}{\sqrt{2t}}}^\infty e^{-\frac{1}{2}u^2} du \quad \text{is} \quad \sqrt{\frac{\pi}{2}} \frac{e^{-a\sqrt{s}}}{s},$$

where  $a$  is a positive constant, show that

$$T(x, t) = \sqrt{\frac{2}{\pi}} T_0 \int_{\frac{x}{\sqrt{2\kappa t}}}^\infty e^{-\frac{1}{2}u^2} du. \quad (2 \text{ marks})$$

Check that the conditions as  $t \rightarrow 0+$  for  $x > 0$ , at  $x = 0$  for  $t > 0$ , and as  $x \rightarrow \infty$  for  $t > 0$  are satisfied.

(6 marks)

For fixed  $x > 0$ , find

$$\lim_{t \rightarrow \infty} T(x, t). \quad (2 \text{ marks})$$

$$\left[ \begin{array}{l} \text{You may use} \\ \int_{-\infty}^\infty e^{-v^2} dv = \sqrt{\pi}. \end{array} \right]$$

- 3 The function  $y(x)$  satisfies the ordinary differential equation

$$x^2 y'' - 2y = \ln(1+x)$$

in  $0 < x < 1$ , with the boundary conditions  $y = 0$  at  $x = 0$  and at  $x = 1$ .

- (a) By trying  $y = x^n$ , find the independent solutions of

$$x^2 y'' - 2y = 0. \quad (3 \text{ marks})$$

- (b) Given that Green's function,  $G(x; \xi)$ , for the boundary-value problem given at the beginning of the question is continuous at  $x = \xi$ , and that  $\partial G / \partial x$  has a discontinuity of size  $1/\xi^2$  at  $x = \xi$ , show that

$$G(x; \xi) = \begin{cases} \frac{1}{3} (1 - \xi^{-3}) x^2 & 0 \leq x < \xi \\ \frac{1}{3} (x^2 - x^{-1}) & \xi < x \leq 1. \end{cases} \quad (13 \text{ marks})$$

- (c) Use Green's function to show that the solution to the boundary-value problem given at the beginning of the question is

$$y(x) = \frac{1}{3} (x^2 - x^{-1}) \{ (1+x) \ln(1+x) - x \} + \frac{1}{3} x^2 \int_x^1 (1 - \xi^{-3}) \ln(1+\xi) d\xi.$$

(6 marks)

Use this to find  $y'(x)$ , and hence find  $y'(1)$ .

(3 marks)

4 Consider the equation

$$\epsilon x^3 + x^2 - 1 = 0, \quad (2)$$

where  $\epsilon$  is a constant satisfying  $0 < \epsilon \ll 1$ .

(a) The solution to equation (2) can be written as

$$x = \frac{1}{\epsilon} (x_0 + \epsilon x_1 + \epsilon^2 x_2 + \epsilon^3 x_3 + \dots),$$

where  $x_0, x_1, x_2, \dots$  are  $O(1)$  as  $\epsilon \rightarrow 0$ .

Use this expression to show that one of the three solutions to equation (2) is

$$x = 1 - \frac{1}{2}\epsilon + O(\epsilon^2),$$

and to find the other two solutions, correct to order  $\epsilon$  as  $\epsilon \rightarrow 0$ .

**(19 marks)**

(b) Given the rearrangement

$$x = -\frac{1}{\epsilon} \left( 1 - \frac{1}{x^2} \right)$$

of equation (2), use iteration to find the solution close to  $-1/\epsilon$ , correct to order  $\epsilon^3$  as  $\epsilon \rightarrow 0$ .

**(6 marks)**

- 5 The Laplace transform,  $\tilde{f}(s)$ , of a function  $f(t)$  is defined by

$$\tilde{f}(s) = \mathcal{L}\{f(t)\} = \int_0^{\infty} e^{-st} f(t) dt,$$

where  $s$  is real and positive.

- (a) By integrating by parts, show that if  $f$  is  $n$  times differentiable then

$$\tilde{f}(s) = \frac{f(0)}{s} + \frac{f'(0)}{s^2} + \cdots + \frac{f^{(n-1)}(0)}{s^n} + R_n(s), \quad (3)$$

where

$$R_n(s) = \frac{1}{s^n} \int_0^{\infty} e^{-st} f^{(n)}(t) dt. \quad (8 \text{ marks})$$

[ You may assume that  $\lim_{t \rightarrow \infty} \{e^{-st} f^{(m)}(t)\} = 0$  for  $m = 0, 1, \dots, n-1$ . ]

- (b) Use equation (3) to deduce that for  $f(t) = \sin t$

$$\tilde{f}(s) = \frac{1}{s^2} - \frac{1}{s^4} + \frac{1}{s^6} - \frac{1}{s^8} \cdots + R_n(s). \quad (6 \text{ marks})$$

Show that in this case

$$|R_n(s)| \leq \frac{1}{s^{n+1}}$$

and, hence, that the series is an asymptotic expansion as  $s \rightarrow \infty$ .

(6 marks)

Show that the exact result for  $\tilde{f}(s)$  gives the same expansion as  $s \rightarrow \infty$ .

(5 marks)

**End of Question Paper**