Data provided: formula sheet

AER201

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2019–20**

Mathematics for Aerospace Engineers **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

Please leave this exam paper on your desk
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to be completed by student

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1 (i) A bivariate random variable $(X, Y)^T$ has joint probability mass function:

	$p_{X,Y}(x,y)$	$y = 0$	1	2
	0	16/50	4/50	0
x	1	16/50	8/50	1/50
	2	0	4/50	1/50

(a) Find the marginal distributions of X and Y. **(2 marks)**

(b) Find the mean and variance of X. **(3 marks)**

(c) Are X and Y independent? Justify your answer. **(2 marks)**

(ii) A system component may fail in one of two ways: mode I or mode II. Which of the two modes is more likely depends on where in the component's lifecycle the underlying causes lie: manufacture, system assembly or during use. For example, issues at manufacturing predominantly (80% of the time) result in mode II failure. Past records suggest the probabilities of failure in each mode are as follows:

	I	II
Cause lies in Manufacture	0.2	0.8
Cause lies in Assembly	0.4	0.6
Cause lies in Usage	0.75	0.25

Overall, few faults (10%) are attributable to manufacturing problems, 20% can be considered due to incorrect assembly, while the remainder are due to inappropriate usage.

A forensic examination of a broken component implicated in an industrial accident claim finds that it experienced a mode I failure. What is the probability that the fault lies with its manufacture? **(5 marks)**

(iii) A single water filter in a filtration system has a lifetime, X, which is Exponentially distributed with mean of 3 days. The filter is replaced as soon as it fails with an identical filter (assumed independent). Define the total lifetime of 100 such filters as

$$S_{100} = \sum_{i=1}^{100} X_i$$

where each X_i is independently and identically distributed as X.

(a) What is the approximate distribution of S_{100}? **(5 marks)**

(b) Write down an expression for the approximate probability that this total lifetime exceeds one year (365 days) and give a MatLab command which would evaluate this probability. **(3 marks)**

2 Consider the function

$$f(t) = \begin{cases} 1, & (0 \leq t \leq \pi) \\ -1, & (-\pi \leq t < 0) \end{cases}$$

on the interval $[-\pi, \pi]$.

(i) Show that the Fourier series $S[f](t)$ on $[-\pi, \pi]$ is

$$S[f](t) = \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{\sin(2m-1)t}{2m-1}. \qquad (1)$$

(12 marks)

(ii) Find the sum of the infinite series

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

by choosing t suitably in the above Fourier series. **(4 marks)**

(iii) For

$$g(t) = |t| \ \ (-\pi \leq t \leq \pi),$$

find $S[g](t)$ by formally integrating (1) term-by-term with respect to t, or otherwise. **(4 marks)**

3 (i) (a) Let $f(x,y) = x^3 - y^3 - 3x + 12y$. Calculate the partial derivatives

$$f_x,\ f_y,\ f_{xx},\ f_{yx},\ f_{xy}.$$

(2 marks)

(b) Hence, find and classify *all* the critical points of the function $f(x,y)$.

(8 marks)

(ii) Consider the transformation of Cartesian to the polar coordinates

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta. \end{cases}$$

(a) By chain-rule differentiation, show that for any function $f(x,y)$

$$\begin{cases} \dfrac{\partial f}{\partial r} = \cos\theta \dfrac{\partial f}{\partial x} + \sin\theta \dfrac{\partial f}{\partial y}, \\ \dfrac{\partial f}{\partial \theta} = -r\sin\theta \dfrac{\partial f}{\partial x} + r\cos\theta \dfrac{\partial f}{\partial y}. \end{cases}$$

(6 marks)

(b) By (a), show that

$$\left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2.$$

(4 marks)

4 (i) Let $R = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le 1\}$. Calculate

$$\iint_R x e^{xy}\, dA.$$

(5 marks)

(ii) Let $R = \{(x,y) \mid 0 \le x \le 1,\ x^2 \le y \le x\}$. Sketch the region of integration and calculate

$$\iint_R 6x^{-2} y\, dA.$$

(7 marks)

(iii) Let $R = \{(x,y,z) \mid 0 \le x \le 1,\ 0 \le y \le 2,\ 0 \le z \le 2x + 3y\}$. Find the volume of R.

(8 marks)

5 (i) Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field defined by

$$\mathbf{F} \;=\; \left(e^{x+y},\ x^2 + y,\ xz^2 + y^2\right)$$

where $x,\ y,\ z \in \mathbb{R}$.

(a) Calculate **curl F**. **(5 marks)**

(b) Calculate **div F**. **(3 marks)**

(c) Calculate the Laplacian of **F**. **(4 marks)**

(ii) Show that **div curl** $= 0$, that is

$$\nabla \cdot (\nabla \times \mathbf{G}) = 0$$

for all vector fields $\mathbf{G} : \mathbb{R}^3 \to \mathbb{R}^3$. **(4 marks)**

(iii) Let $f(x,y) = x^2y + xy + 1$. Calculate the directional derivative of $f(x,y)$ at $(x,y) = (1,0)$ in the direction $v = (2,1)$. **(4 marks)**

End of Question Paper

AER201 FORMULA SHEET

Standard Probability Distributions:

Name	**Applications**	**Notation**	**pmf** or **pdf**	$E(X)$	$\text{Var}(X)$
Bernoulli trial	Expt with two outcomes. Coins, constituent of more complex distributions. $X \equiv$ no. successes	$Bernoulli(p)$	$p_X(1) = p$, $p_X(0) = 1 - p$ $p\epsilon[0, 1]$	p	$p(1 - p)$
Binomial	$X \equiv$ no. successes in n ind. Bernoulli trials Sampling with replacement	$Bin(n, p)$	$p_X(x) = \frac{n!}{x!(n-x)!} p^x (1-p)^{n-x}$ $x = 0, 1, 2, ..., n$ $p\epsilon[0, 1]$	np	$np(1 - p)$
Geometric	$X \equiv$ total no. of trials until 1st success in sequence of ind. Bernoulli trials Waiting times	Geo(p)	$p_X(x) = (1-p)^{x-1}p$ $x = 1, 2, ...$ $p\epsilon[0, 1]$	$\frac{1}{p}$	$\frac{1-p}{p^2}$
Poisson	Counting events occurring 'at random' in space or time	$Po(\lambda)$	$p_X(x) = \frac{e^{-\lambda}\lambda^x}{x!}$ $x = 0, 1, 2, ...$ $\lambda > 0$	λ	λ
Multinomial	Generalization of Binomial to > 2 categories	$multinomial(n; p_1, ..., p_k)$	$p_X(x_1, \ldots, x_k) =$ $\frac{n!}{x_1!x_2!\cdots x_k!} p_1^{x_1} p_2^{x_2} \ldots p_k^{x_k}$		
Uniform	Rounding errors $Un(-\frac{1}{2}, \frac{1}{2})$	$Un(a, b)$	$f_X(x) = \dfrac{1}{b-a}$ $x \in [a, b]$ $a < b$	$\frac{a+b}{2}$	$\frac{(b-a)^2}{12}$
Exponential	Lifetimes of non-ageing items	$Exp(\lambda)$	$f_X(x) = \lambda e^{-\lambda x}$ $x > 0$ $\lambda > 0$	$\dfrac{1}{\lambda}$	$\dfrac{1}{\lambda^2}$
Normal	Empirically, and theoretically via CLT, a good model in many situations	$N(\mu, \sigma^2)$	$f_X(x) =$ $\dfrac{1}{\sqrt{2\pi\sigma^2}} \exp\left\{-\frac{1}{2\sigma^2}(x-\mu)^2\right\}$ $x \in (-\infty, \infty)$	μ	σ^2
Multivariate Normal	Empirically a good model in many situations	$N_k(\boldsymbol{\mu}, \boldsymbol{\Sigma})$	$f_X(x_1, x_2, ..., x_k) =$ $\frac{1}{\sqrt{(2\pi)^k\lvert\Sigma\rvert}} e^{\frac{-(\mathbf{X}-\boldsymbol{\mu})^T\boldsymbol{\Sigma}^{-1}(\mathbf{X}-\boldsymbol{\mu})}{2}}$	$\boldsymbol{\mu}$	$\boldsymbol{\Sigma}$

Bayes' Theorem:

Suppose we have two events E and F within a sample space S, then

$$P(E|F) = \frac{P(F|E)P(E)}{P(F)}$$

Central Limit Theorem:

Let $X_1, X_2, \ldots, X_n$ be a sequence of i.i.d random variables, each with mean μ and variance σ^2, then for large n we have, approximately,

$$\bar{X}(n) \sim N\left(\mu, \frac{\sigma^2}{n}\right)$$

or, equivalently,

$$\Sigma_{i=1}^{n} X_i \sim N\left(n\mu, n\sigma^2\right)$$

Laplace transform:

The Laplace transform of a function $f(t)$ is given by:

$$\mathcal{L}\{f(t)\}(s) := \int_0^{\infty} e^{-st} f(t) dt.$$

Properties of the Laplace transform: $\mathcal{L}\{f(t)\} = F(s)$ in the following table.

$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$	linearity
$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$	differentiation w.r.t. t
$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$	second differentiation w.r.t. t
$\mathcal{L}\{e^{-kt}f(t)\} = F(k+s)$	frequency shift
$\mathcal{L}\{f(t-a)H(t-a)\} = e^{-as}F(s) \quad$ (for $a > 0$)	time shift
$\mathcal{L}\{f(at)\} = \frac{1}{a}F(\frac{s}{a}) \quad$ (for $a > 0$)	scaling
$\mathcal{L}\{f * g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$ (for $f(t), g(t)$ causal)	convolution

Table of standard Laplace transforms:

$f(t)$	$\mathcal{L}\{f(t)\}(s)$	Region of validity
$t^n \quad$ (for $n \geq 0$)	$\frac{n!}{s^{n+1}}$	$Re(s) > 0$
$\sin(kt)$	$\frac{k}{s^2+k^2}$	$Re(s) > 0$
$\cos(kt)$	$\frac{s}{s^2+k^2}$	$Re(s) > 0$
$H(t-T) \quad$ (for $T \geq 0$)	$\frac{e^{-sT}}{s}$	$Re(s) > 0$
$\delta(t-T) \quad$ (for $T \geq 0$)	e^{-sT}	$s \in \mathbb{C}$

Fourier transform:

The Fourier transform and inverse Fourier transforms are given by:

$$\mathcal{F}\{f(t)\}(\omega) = F(\omega) := \int_{-\infty}^{\infty} f(t)e^{-j\omega t} dt, \qquad f(t) = \mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t} d\omega.$$

Properties of the Fourier transform: $\mathcal{F}\{f(t)\} = F(\omega)$ in the following table:

$\mathcal{F}\{e^{j\theta t}f(t)\} = F(\omega-\theta)$	frequency shift
$\mathcal{F}\{f(t-T)\} = e^{-j\omega T}F(\omega)$	time shift
$\mathcal{F}\{f^{(n)}(t)\} = (j\omega)^n F(\omega)$	differentiation
$\mathcal{F}\{F(t)\} = 2\pi f(-\omega)$	symmetry
$\mathcal{F}\{f(at)\} = \frac{1}{\lvert a\rvert}F(\frac{\omega}{a})$	scaling
$\mathcal{F}\{f * g(t)\} = \mathcal{F}\{f(t)\}\mathcal{F}\{g(t)\}$	convolution

Table of standard Fourier transforms:

$f(t)$	$\mathcal{F}\{f(t)\}(\omega)$
$e^{-a\lvert t\rvert}$ (for $a > 0$)	$\frac{2a}{a^2+\omega^2}$
$\text{rect}_T(t)$	$\text{sinc}(\frac{T\omega}{2})$
1	$2\pi\delta(\omega)$

Fourier series:

The Fourier series of a periodic function $f(t)$ with fundamental period T is given by

$$S[f] = \frac{a_0}{2} + \sum_{n=1}^{\infty}\Big(a_n\cos(\omega_n t) + b_n\sin(\omega_n t)\Big)$$

where

$$\omega_n = \frac{2\pi n}{T}, \qquad a_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\cos(\omega_n t)dt, \qquad b_n = \frac{2}{T}\int_{-T/2}^{T/2} f(t)\sin(\omega_n t)dt.$$

Coordinate systems:

Cylindrical polar coordinates

$$(x,y,z) = (r\cos(\theta), r\sin(\theta), z)$$

$$(r,\theta,z) = \Big(\sqrt{x^2+y^2}, \arctan(\tfrac{y}{x}), z\Big)$$

$$dV = r dr d\theta dz.$$

Spherical polar coordinates

$$(x,y,z) = (\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi))$$

$$(\rho,\theta,\phi) = \Big(\sqrt{x^2+y^2+z^2}, \arctan(\tfrac{y}{x}), \arccos(\tfrac{z}{\rho})\Big)$$

$$dV = \rho^2\sin(\phi) d\rho d\phi d\theta.$$