The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Spring Exam Period 2019-2020**

FOUNDATION YEAR CORE MATHEMATICS **Submission by: 13.00 BST, Wednesday, 3rd June 2020**

This is an open book exam.
*Answer **all** questions. The allocation of marks is shown in brackets.*
You will have 48 hours to submit the paper, however, it should take approximately three and a half hours to answer all questions.
By uploading your solutions you declare that your submission consists entirely of your own work, that any use of sources or tools other than material provided for this module is cited and acknowledged and that no unfair means have been used.
*This exam paper has two sections. Section A consists of multiple choice questions **which must be answered on the pre-circulated MCQ answer sheet.** Answers to Section B must be written on clean sides of paper having a 2.5 cm margin on the right hand side and show clearly your working.*
Total marks: 90.

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student

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Section A

Each question or incomplete statement in this section is followed by four possible options of which exactly one is correct. Write the corresponding letter on the pre-circulated MCQ answer sheet. Total marks for this section: ***(30 marks)***

A1 Let $q = 2p + 1$. The statement that $\sqrt[q]{-\frac{1}{6}} \in \mathbb{R}$ is true if and only if

A. p is even; **B.** p is odd; **C.** $p \neq 0$; **D.** p is an integer.

A2 Let $y,\ z > 0$. The expression $z^{-y^2} z^{-y \log_z \left(\frac{1}{z}\right)^y} =$

A. z^{1-y^2}; **B.** -1; **C.** 1; **D.** z^{y^2-1}.

A3 Let $\gamma > 0$. The expression $\dfrac{\log_{25} \gamma}{\log_{625} \gamma} =$

A. $\dfrac{1}{125}$; **B.** 2; **C.** $\log_{600} \gamma$; **D.** $-\dfrac{1}{2}$.

A4 For $d > 1$, as $u \to 0$, $\log_d u \to$

A. $-\infty$; **B.** d;

C. 0; **D.** ∞.

A5 Let $S_A = \displaystyle\sum_{a=1}^{A} ra,\ |r| < 1,\ r \neq 0$. Then S_A is equal to:

A. $\dfrac{a}{2}\left(2A + n(r-1)\right)$; **B.** $\dfrac{r}{2}(A+1)A$;

C. $A!$; **D.** $\dfrac{a}{1-r}$.

A6 The result, S_A of the series given in question **A5** above is also equal to:

A. $\binom{A-1}{A+1}r$; **B.** $(A+1)!Ar$; **C.** $A(A-1)!r$; **D.** $\binom{A+1}{2}r$.

A7 The series

$$\sum_{t=1}^{A} dcba^{t-3},$$

(where d, c, b are non-zero constants) can be expressed as

A. $\displaystyle\sum_{t=0}^{A-1} dcba^{t-3}$; **B.** $\displaystyle dcba^3 \sum_{t=0}^{A-1} a^t$;

C. $\displaystyle dcba^{-3} \sum_{t=0}^{A} a^t$; **D.** $\displaystyle dcb \sum_{t=-2}^{A-3} a^t$

A8 Which one of the results below corresponds to $\cos^2\left(\frac{\pi}{8}\right) - \sin^2\left(\frac{\pi}{8}\right)$?

A. $2\sqrt{2}$; **B.** $\frac{2}{\sqrt{2}}$; **C.** $\sqrt{2}$; **D.** $\frac{\sqrt{2}}{2}$.

A9 Given that $\sin^2\mu + \cos^2\mu = 1$ for all $\mu \in \mathbb{R}$, which one of the following statements is true for all $0 < \mu < \frac{\pi}{2}$?

A. $\operatorname{cosec}\mu = \sin\mu - \cot\mu\cos\mu$; **B.** $\operatorname{cosec}2\mu = \frac{1}{2}\left(\tan\mu + \tan^{-1}\mu\right)$;

C. $\operatorname{cosec}\mu = \cot\mu\cos\mu - \sin\mu$. **D.** $\operatorname{cosec}2\mu = \frac{1}{2}\left(\cot\mu + (\cot\mu)^{-1}\right)$

A10 An unbiased tetrahedral die has faces labelled by the first four double-digit prime numbers. Let R correspond to rolling the die and obtaining a number $m \geq 18$. Given this information, which one of the following statements are true?

A. $\mathrm{P}(R) = 3$; **B.** $\mathrm{P}(\overline{R}) = 4$; **C.** $\mathrm{P}(\overline{R}) = \frac{3}{4}$; **D.** $\mathrm{P}(R) = 0.75$.

A11 Given an event E, which one of the following statements is always meaningful?

A. $\mathrm{P}(E) = 3$; **B.** $\mathrm{P}(\overline{E}) = \frac{1}{100}$;

C. $\mathrm{P}(E) = 1.1$; **D.** $\mathrm{P}(\overline{E}) = -\frac{1}{100}$.

A12 Which one of the following expressions is undetermined?

A. $\frac{0}{x},\ (x \neq 0)$; **B.** $\frac{0}{0}$; **C.** $\frac{x}{x},\ (x \neq 0)$; **D.** $\frac{x}{0},\ (x \neq 0)$.

A13 Which one of the following expressions has a result that is undefined over the reals?

A. $\left(\frac{1}{7}\right)^6$; **B.** $6^{-\frac{1}{7}}$; **C.** $7^{-\frac{1}{6}}$; **D.** $\left(-\frac{1}{7}\right)^{\frac{1}{6}}$.

A14 If $\boldsymbol{v} = \boldsymbol{i} + \frac{1}{100}\boldsymbol{j},\ \boldsymbol{w} = \boldsymbol{i} + \boldsymbol{j}$ and $l \in \mathbb{R}$, then $\sum_{p=1}^{\infty} l\,(\boldsymbol{v} \bullet \boldsymbol{w})^p$

A. is meaningless; **B.** converges;

C. diverges; **D.** is a finite series.

A15 If $t(f) = \sum_{q=1}^{7} \binom{7}{q} (\boldsymbol{bv})^{q-7} f^q$ then $t(f)$

A. is meaningless;

B. is a binomial expansion of $(\boldsymbol{bv} + f)^5$;

C. is a finite geometric series;

D. $= (\boldsymbol{bv} + f)^5 - (\boldsymbol{bv})^{-5}$.

A16 The derivative of $y = x^3 + \frac{4}{x^{\frac{3}{16}}} + \sqrt[5]{x} + 2 + \pi^2 e^3$ is $y' =$

A. $3x^2 - \frac{3}{4}x^{\frac{19}{16}} + \frac{1}{5x^{\frac{4}{5}}} + 3\pi^2 e^2$;

B. $3x^2 - \frac{3}{4}x^{\frac{19}{16}} + \frac{1}{5x^{\frac{4}{5}}}$;

C. $3x^2 - \frac{3}{4}x^{\frac{19}{16}} - \frac{1}{5x^{\frac{4}{5}}}$;

D. $3x^2 - \frac{3}{4}x^{\frac{19}{16}} + \frac{1}{5x^{\frac{4}{5}}} + 3\pi^2 e^{3x}$.

A17 The derivative of $y = 2e^{5\sin x}$ is $y' =$

A. $10\cos(x)e^{5\sin x}$;

B. $10\sin(x)e^{5\sin x}$;

C. $\frac{2}{5}\cos(x)e^{5\sin x}$;

D. $10\sin(x)e^{(5\sin x)-1}$.

A18 The derivative of $y = (\ln(3+x^2)(\tan^{-1}(x))$ is $y' =$

A. $\frac{\tan^{-1}(x)}{3+x^2} + \frac{\ln(3+x^2)}{x^2+1}$;

B. $\frac{2x\tan^{-1}(x)}{3+x^2} + \frac{\ln(3+x^2)}{x^2+1}$;

C. $\frac{2x\tan^{-1}(x)}{3+x^2} - \frac{\ln(3+x^2)}{x^2+1}$;

D. $\frac{2x}{(3+x^2)(x^2+1)}$.

A19 The derivative of $y = \frac{x^3}{\cos(x)}$ is $y' =$

A. $\frac{x^2(3\cos(x) + x\sin(x))}{\cos^2 x}$;

B. $\frac{x^2(3\cos(x) - x\sin(x))}{\cos^2 x}$;

C. $\frac{x^3\sin(x) - 3x^2\cos(x)}{\cos^2 x}$;

D. $\frac{x^2(3\cos x + x\sin x)}{x^6}$.

A20 $\displaystyle\int \left(6x^5 - \frac{6}{x^{\frac{2}{5}}} - \frac{1}{x} + \pi\right) dx =$

A. $x^6 - 10x^{\frac{3}{5}} - \ln x + \pi x + c$;

B. $x^6 - 10x^{\frac{3}{5}} - \ln|x| + \pi x + c$;

C. $x^6 - 10x^{\frac{3}{5}} - \ln|x| + c$;

D. $30x^6 - 10x^{\frac{3}{5}} - \ln|x| + \pi x + c$.

A21 The equations of the tangent, $t(x)$, and the normal, $n(x)$, of $y = \ln(x - e)$ at $x = 2e$ are

A. $t(x) = \frac{1}{e}x - 1,$ and $n(x) = -ex + 1 + 2e^2$;

B. $t(x) = \frac{1}{e}x,$ and $n(x) = -ex - 1$;

C. $t(x) = -\frac{1}{e}x - 1,$ and $n(x) = ex$;

D. $t(x) = x - 1,$ and $n(x) = -x + 1$.

A22 $\displaystyle\int x^2 \sin(2x)\, dx =$

A. $\frac{1}{2}\left(\frac{1}{2}\cos(2x) + x\sin(2x) - x^2\cos(2x)\right) + c$;

B. $\frac{1}{2}\left(\frac{1}{2}\cos(2x) + x\sin(2x) - x^2\cos(2x)\right)$;

C. $-\frac{1}{6}x^3\cos(2x) + c$;

D. $2\left(2\cos(2x) + x\sin(2x) - x^2\cos(2x)\right) + c$.

A23 $\displaystyle\int \frac{3x}{x^2+1} e^{\ln(x^2+1)}\, dx =$

A. $\ln\left|x^2 + 1\right| + c$;

B. $e^{\ln(x^2+1)} + c$;

C. $\frac{3}{2}x^2 + c$;

D. $\frac{3}{2}e^{\ln(x^2+1)}$.

A24 $\int_{-1}^{2} x^3 - x^2 \, dx =$

A. $\frac{5}{4}$; **B.** $\frac{3}{4}$; **C.** $-\frac{5}{4}$; **D.** $-\frac{3}{4}$.

A25 $\int_{0}^{1} e^{4x+1} \, dx =$

A. $e(e^4 + 1)$; **B.** πe^2; **C.** $\frac{1}{5}(e^5 - e)$; **D.** $\frac{1}{4}e(e^4 - 1)$.

A26 Let $f(x)$ be an even function and a be a real number. Assume that $\int_{-a}^{a} f(x) \, dx$ exists. Then $\int_{-a}^{a} f(x) \, dx =$

A. $2\int_{0}^{a} f(x) \, dx$; **B.** $\int_{0}^{2a} f(x) \, dx$; **C.** 0; **D.** 1.

A27 If $x = \frac{t^2}{(t-1)^2}$ and $y = \frac{t + 2t^2}{t - 1}$ then $\frac{dy}{dx} =$

A. $-\frac{(1+2t)(t-1)}{t}$ for $t \neq 0$;

B. $\frac{(1-t)\left(t - \frac{1}{2}(2+\sqrt{6})\right)\left(t - \frac{1}{2}(2-\sqrt{6})\right)}{t}$ for $t \neq 0$;

C. $-\frac{2t}{2t^3 - 6t^2 + 3t + 1}$ for $2t^3 - 6t^2 + 3t + 1 \neq 0$;

D. $-\frac{2t}{2t^3 - 6t^2 + 3t + 1}$ for $2t^3 - 6t^2 + 3t + 1 \neq 0$.

A28 The area enclosed by the curve $y = x^2 - 5x + 6$, the x-axis and the lines $x = \frac{3}{2}$ and $x = \frac{5}{2}$ is

A. $\frac{1}{12}$; **B.** e; **C.** $\frac{1}{4}$; **D.** $\frac{29}{19}$.

A29 If $y = 3^{4x}$ then $\dfrac{dy}{dx}$ is

A. 3^{4x-1};
B. $4 \times 3^{4x-1}$;
C. $(\ln 12)3^{4x}$;
D. $(4\ln 3)3^{4x}$.

A30 $\displaystyle\int \frac{4x^{13} - 4x^{12} + 5x^{11} - 4x^{10} + x^9 + 12x^5 + 11x^4 - 15x^3 + 17x^2 - 13x + 4}{x^2(2x-1)^2(x^2+1)}\, dx =$

A. $\dfrac{1}{8}x^8 + 3\ln|x| - \dfrac{4}{x} - 2\ln|2x-1| - \dfrac{3}{2(2x-1)} + \ln(x^2+1) + 2\tan^{-1}(x) + c$;
B. $\dfrac{1}{8}x^8 + 3\ln|x| - \dfrac{4}{x} - 2\ln|2x-1| - \dfrac{3}{2(2x-1)} + \ln(x^2+1) + 2\tan^{-1}(x)$;
C. $3\ln|x| - \dfrac{4}{x} - 2\ln|2x-1| - \dfrac{3}{2(2x-1)} + \ln(x^2+1) + 2\tan^{-1}(x) + c$;
D. $\dfrac{1}{8}x^8 + 3\ln(x) - \dfrac{4}{x} - 2\ln(2x-1) - \dfrac{3}{2(2x-1)} + \ln(x^2+1) + 2\tan^{-1}(x) + c$.

Section B

Give your full solution on clean sides of paper having a 2.5 cm margin on the right hand side for each of the following questions. You need to clearly show your workings. Total marks for this section: ***(60 marks)***

B1 (i) Solve the following equation for a, where $a > 0$ and $0 < b < 1$.

$$\log_b(1-a) + \log_b(1+a) = 1.$$

(3 marks)

(ii) By exponentiating and taking logarithms of $p = \ln q$, show that

$$p = \frac{\log_p q}{\log_p e}.$$

(3 marks)

B2 (i) Find the coefficient of x^3 in the binomial series expansion of

$$\left(6^{-\frac{1}{3}}x + 14^{-\frac{1}{6}}\right)^9.$$

Note: You may not use Pascal's triangle to reach your solution.

(3 marks)

(ii) Show that the sum of the first n terms of an arithmetic sequence, S_n can be written as

$$S_n = \frac{n}{2}\left(a + (n-1)d\right),$$

where a is the first term of the sequence and d is the constant difference between successive terms.

Hint: Start from the relation

$$S_n = a + (a+d) + (a+2d) + \ldots + (a+(n-1)d).$$

(3 marks)

B3 (i) The function $u(z) = \frac{1}{2}(\cos z + \sin z)$ can be expressed as $K\sin(z+\gamma)$.

Find K and γ, where $0 < \gamma < \frac{\pi}{2}$.

Formulae such as e.g. $a\cos z + b\sin z = (\sqrt{a^2+b^2})\sin(z + \tan^{-1}(a/b))$ may not be used directly here. You need to show your result using steps in the lectures, explicitly. ***(3 marks)***

(ii) Given the compound angle formulae for sine and cosine, derive the compound angle formula for $\tan(\alpha+\beta)$.

(3 marks)

B4 (i) The probability distribution of a discrete random variable, T, is as follows:

t	1	2	3
P_t	0.05	P_2	0.35

where P_t denotes the probability that $T = t$.

Find $\mathrm{E}\left(T^2\right)$, simplifying your answer as far as possible. ***(3 marks)***

(ii) Write the standard formula for the expression

$$\mathrm{P}(X|X \cap Y)$$

and evaluate this. ***(3 marks)***

B5 (i) Let $\boldsymbol{f} = 4\boldsymbol{i} - 3\boldsymbol{j} + 2\boldsymbol{k}$ and $\boldsymbol{g} = 2\sigma\boldsymbol{i} + \boldsymbol{j} + \boldsymbol{k}$.

Find σ when is $\boldsymbol{f}$ is perpendicular to $\boldsymbol{g}$. ***(3 marks)***

(ii) Given that the scalar product distributes i.e.

$$\boldsymbol{u} \bullet (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{u} \bullet \boldsymbol{v} + \boldsymbol{u} \bullet \boldsymbol{w},$$

for any vector $\boldsymbol{u}$, $\boldsymbol{v}$, $\boldsymbol{w}$, show that

$$(\boldsymbol{u} + \boldsymbol{v}) \bullet (\boldsymbol{u} - \boldsymbol{v}) = |\boldsymbol{u}|^2 - |\boldsymbol{v}|^2.$$

(3 marks)

B6 Find the stationary points of

$$f(x) = \frac{1}{7}x^7 - \frac{1}{3}x^6 + \frac{1}{2}x^4 - \frac{1}{3}x^3 + e$$

and determine their nature. ***(10 marks)***

B7 Find $\displaystyle\int_2^3 \frac{x^6 - x^5 - x^2 + x - 1}{x^2(x^2-1)}\,dx.$ ***(12 marks)***

B8 Find the *area* enclosed by $y = e^{x-3}$, $y = \ln(x^2 + 1)$, $x = 1$ and $x = e - 1$.
Your answer should be exact.
You might want to know that $\ln(2) \approx 0.7$ and $\ln((e-1)^2 + 1) \approx 1.4$. ***(8 marks)***

End of Question Paper