The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2019–20**

Mathematics Core 1 **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

This exam paper has two sections. Section A consists of multiple choice questions which must be answered on the exam paper itself.

Answers to Section B must be written on the answer booklet provided.

Total marks: ***50***

Please leave this exam paper on your desk
Do not remove it from the hall

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to be completed by student

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Section A:

Each question or incomplete statement in this section is followed by four possible options of which exactly one is correct. Mark clearly the correct answer on the question paper. ***(24 marks)***

A1 Let $A = \{1, 2\}$ and $B = \{2, 3\}$. Which of the following options is a subset of $A \times B$?

A. $(2, 1)$ **B.** $\{2, 2\}$ **C.** $\{(2, 2)\}$ **D.** $(2, 2)$

A2 Let $A = \{1, 2\}$ and $B = \{2, 3\}$. What is the cardinality of $(A \times B) - (B \times A)$?

A. 1 **B.** 2 **C.** 3 **D.** 4

A3 The number of subsets of $\{1, 2, 3, 4, 5, 6, 7, 8\}$ is

A. 2^8 **B.** 8^2 **C.** 8^8 **D.** $(8 \times 8)^4$

A4 How many different functions from $\{1, 2, 3\}$ to $\{1, 2\}$ are there?

A. 6 **B.** 8 **C.** 9 **D.** $\binom{3}{2}$

A5 How many different anagrams does `SHEFFIELD` have?

A. 9^7 **B.** $2 \times 2 \times 7!$ **C.** $\binom{9!}{7!}$ **D.** $\dfrac{9!}{2 \times 2}$

A6 If $r \neq 1$ then $r^3 + r^6 + r^9 + r^{12} + r^{15}$ is equal to

A. $\dfrac{r^3(1 + r^{15})}{1 + r^3}$ **B.** $\dfrac{r^3(1 - r^{15})}{1 - r^3}$ **C.** $\dfrac{r^3(1 + r^3)^5}{1 + r^5}$ **D.** $\dfrac{1 - r^{15}}{1 - r^3}$

A7 If $0 < \theta < \dfrac{\pi}{2}$ and $\cos\theta = \dfrac{3}{5}$ then $\sin(2\theta) =$

A. $\dfrac{6}{25}$ **B.** $\dfrac{12}{25}$ **C.** $\dfrac{18}{25}$ **D.** $\dfrac{24}{25}$

A8 If $0 < x, y < \frac{\pi}{2}$ then $\dfrac{\cos(x+y)}{\cos x \sin y} =$

A. $\cot y - \tan x$ **B.** $\cot y + \tan x$ **C.** $\cot x - \tan y$ **D.** $\cot x + \tan y$

A9 $\lim\limits_{x\to 0} \dfrac{e^{3x}-1}{x} =$

A. 3 **B.** 2 **C.** 0 **D.** ∞

A10 If $f : \mathbb{R} \to \mathbb{R}$ is given by

$$f(x) = \begin{cases} 2x^2+1, & x \neq 0; \\ 0, & x = 0; \end{cases}$$

then $\lim\limits_{x\to 0} f(x) =$

A. 1 **B.** 0 **C.** -1 **D.** does not exist

A11 If f is differentiable at 1 with derivative 1 then $\lim\limits_{h\to 0} \dfrac{f(1-h)-f(1)}{h} =$

A. 0 **B.** 1 **C.** -1 **D.** not enough data

A12 The tangent to the curve $x^3 + y^3 = 2$ at the point $(1, 1)$ has slope

A. 1 **B.** 0 **C.** -1 **D.** ∞

A13 If $\cos x = e^y$ then $\dfrac{dy}{dx} =$

A. $\tan x$ **B.** $\cot x$ **C.** $-\tan x$ **D.** $-\cot x$

A14 $\displaystyle\int_0^1 \frac{x^{19}}{1+x^{20}}\,dx =$

A. 0 **B.** $\dfrac{19}{20}$ **C.** $\ln(2^{19})$ **D.** $\dfrac{1}{20}\ln(2)$

A15 Let $f : \mathbb{R} \to \mathbb{R}$ be the function given by

$$f(x) = \begin{cases} 0, & \text{when } x \leq 0; \\ 1, & \text{when } 0 < x < 1; \\ x, & \text{when } 1 \leq x. \end{cases}$$

Then $\displaystyle\int_{-\infty}^{3} f(x)\,dx =$

A. ∞ **B.** 4 **C.** 5 **D.** $\dfrac{7}{2}$

A16 $\displaystyle\int_{-\infty}^{\infty} \frac{1}{1+x^2}\,dx =$

A. π **B.** $\dfrac{\pi}{2}$ **C.** $\dfrac{\pi}{3}$ **D.** $\dfrac{\pi}{4}$

A17 The real part of $\dfrac{1+i}{2-i}$ is

A. $\dfrac{1}{2}$ **B.** $\dfrac{1}{5}$ **C.** $\dfrac{2}{5}$ **D.** $\dfrac{3}{5}$

A18 For which choice of n from the options below is $i^n = i$?

A. 2018 **B.** 2019 **C.** 2020 **D.** 2021

A19 The area of the region $\{z \in \mathbb{C} \ : \ |z - i| \leq 2\}$ is

A. 4 **B.** 4π **C.** $i^2\pi + 4\pi$ **D.** $4\pi i$

A20 If $|z| = 2$ and $|w| = 3$ then $\dfrac{|w^2|}{2 + |z^4|} =$

A. 1 **B.** 2 **C.** $\frac{1}{2}$ **D.** $\frac{1}{9}$

A21 If $\arg(z) = \dfrac{\pi}{100}$ and $|z| = 2^{1/50}$ then $z^{50} =$

A. 1 **B.** 2 **C.** i **D.** $2i$

A22 The function $f : \mathbb{R} \to \mathbb{R}$ satisfies

$$\lim_{h \to 0} \frac{f(3+h) - f(3)}{h} = 3.$$

Here are two statements concerning the behaviour of f at 3:

(I) f is continuous at $x = 3$.

(II) f is differentiable at $x = 3$.

Which of the above statments are true?

A. neither **B.** both **C.** **(I)** only **D.** **(II)** only

A23 Let $f, g : \mathbb{R} \to \mathbb{R}$ be the functions defined by $f(x) = 2x + 1$ and $g(x) = x^2$. Then the set of real numbers where $f \circ g$ and $g \circ f$ have the same image is

A. 0 **B.** 2 **C.** $\{0\}$ **D.** $\{0, -2\}$

A24 If $y = (x^2 + 1)^{(2-3x)}$ then $\dfrac{dy}{dx}$ at $x = 1$ is

A. $-\dfrac{\ln(8e)}{2}$ **B.** $-\dfrac{\ln 8}{2}$ **C.** $-\ln(8e)$ **D.** $-\dfrac{1}{8}$

Section B:

Full credit will only be awarded to clearly presented and logically coherent solutions.

B1 Find the general solution to the differential equation

$$\frac{d^2y}{dt^2} - 2\frac{dy}{dt} + y = e^{2t}$$

where y is a function of t. ***(3 marks)***

B2 Use induction to prove that $n^3 - n$ is a multiple of 3 for all $n \in \mathbb{N}$. ***(3 marks)***

B3 Determine constants a, b so that the function $f : \mathbb{R} \to \mathbb{R}$ given by

$$f(x) := \begin{cases} x^2 + 1, & \text{if } x \leq 0 \\ ax + b, & \text{if } 0 < x \end{cases}$$

is differentiable at $x = 0$. ***(4 marks)***

B4 Assuming the Maclaurin series expansion of $\dfrac{1}{1+x}$, derive the Maclaurin series expansion of $\ln(1 + x)$. Also, write down—without proof—the Macluarin series expansion of $\tan^{-1} x$. ***(3 marks)***

Assume that the above Macluarin series expansions are valid for $|x| < 1$. Evaluate

$$\lim_{x \to 0} \frac{\ln(1+x)\tan^{-1}(x) - x^2 + \frac{1}{2}x^3}{x^5}.$$

(3 marks)

B5 Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function with $f(0) = 110$. Evaluate the limit

$$\lim_{x \to 0} \frac{\int_0^{x^2} f(\sin t)\, dt}{\sin(x^2)}.$$

(3 marks)

B6 If $f : [0, 1] \to \mathbb{R}$ is a continuous function, what is the limit

$$\lim_{n\to\infty} \frac{1}{n}\left(f\left(\frac{1}{n}\right) + f\left(\frac{2}{n}\right) + \cdots + f\left(\frac{n}{n}\right)\right)$$

according to Riemann's Theorem? ***(1 mark)***

(i) Evaluate $\lim_{n\to\infty} \dfrac{\sqrt{1} + \sqrt{2} + \cdots + \sqrt{n}}{n\sqrt{n}}$. ***(2 marks)***

(ii) Calculate $\displaystyle\int_0^1 \ln(1 + x)dx$, and determine the limit

$$\lim_{n\to\infty} \left(\frac{(2n)!}{n!n^n}\right)^{1/n}.$$

(4 marks)

End of Question Paper