The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS | **Autumn Semester 2019–20**

Advanced Calculus and Linear Algebra | **2 hours 30 minutes**

Attempt all the questions. The allocation of marks is shown in brackets. There is a total of 100 marks.

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1 (i) Compute the integral $\int_0^2 \int_0^{2\pi} \int_0^{z/2} \rho \, d\rho \, d\theta \, dz$. This integral in cylindrical polar coordinates represents the volume of a certain cone. What is the height of the cone, and what is the radius of the circular disc bounding it on top? ***(5 marks)***

(ii) Find the coordinates of the centroid of the cone referred to above, by computing the average value of z over the region. Is the fraction of the volume of the cone below its centroid greater than, equal to, or less than a half? [Hint: use a similarity/scaling argument.] ***(8 marks)***

2 (i) Compute the volume of the parallelopiped with edges $\begin{pmatrix}1\\3\\1\end{pmatrix}, \begin{pmatrix}2\\2\\1\end{pmatrix}, \begin{pmatrix}3\\1\\2\end{pmatrix}$. ***(2 marks)***

(ii) Compute the product AB, where

$$A = \begin{pmatrix}1 & 2 & 3\\3 & 2 & 1\\1 & 1 & 2\end{pmatrix} \text{ and } B = \begin{pmatrix}-3 & 1 & 4\\5 & 1 & -8\\-1 & -1 & 4\end{pmatrix}.$$

From this, deduce what must be the volume of the parallelopiped with edges

$$\begin{pmatrix}-3\\5\\-1\end{pmatrix}, \begin{pmatrix}1\\1\\-1\end{pmatrix}, \begin{pmatrix}4\\-8\\4\end{pmatrix}.$$

[Hint: use multiplicativity of determinants.] ***(4 marks)***

(iii) Using the inverse of a matrix, find α, β, γ such that

$$\alpha \begin{pmatrix}1\\3\\1\end{pmatrix} + \beta \begin{pmatrix}2\\2\\1\end{pmatrix} + \gamma \begin{pmatrix}3\\1\\2\end{pmatrix} = \begin{pmatrix}1\\1\\1\end{pmatrix}.$$

[Hint: make use of your product calculation in part (ii).] ***(4 marks)***

3 Let A be the matrix $\begin{pmatrix} 1 & 3 & 1 \\ 2 & 2 & 1 \\ 3 & 1 & 1 \end{pmatrix}$, and let $\ell_A : \mathbb{R}^3 \to \mathbb{R}^3$ be the linear map given by $\ell_A(\mathbf{x}) := A\mathbf{x}$ for all $\mathbf{x} \in \mathbb{R}^3$.

(i) Find a linear dependence relation between the vectors

$$\begin{pmatrix} 1 \\ 2 \\ 3 \end{pmatrix}, \begin{pmatrix} 3 \\ 2 \\ 1 \end{pmatrix} \text{ and } \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}.$$

[Hint: try adding two of them together.] Use this to determine the rank of the matrix A.

Find bases for $\ker(\ell_A)$ (the null-space of A), and $\operatorname{im}(\ell_A)$ (the column space of A). ***(7 marks)***

(ii) Describe $\operatorname{im}(\ell_A)$ implicitly, by an equation $ax + by + cz = 0$. *Hence* show that $\begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix} \in \operatorname{im}(\ell_A)$. Also express $\begin{pmatrix} 3 \\ 5 \\ 7 \end{pmatrix}$ as a linear combination of the basis for $\operatorname{im}(\ell_A)$ that you produced in (i). ***(5 marks)***

(iii) Write down matrices $B \in M_{3,1}(\mathbb{R})$ and $C \in M_{1,3}(\mathbb{R})$ such that $\ker(\ell_A) = \operatorname{im}(\ell_B)$ and $\operatorname{im}(\ell_A) = \ker(\ell_C)$. ***(2 marks)***

4 Define $F : \mathbb{R}^3 \to \mathbb{R}$ by $F\left(\begin{pmatrix} x \\ y \\ z \end{pmatrix}\right) = x^2 + y^2 - z^2 - 1$.

Consider also the map $\mathbf{r} : \mathbb{R} \to \mathbb{R}^3$ defined by $\mathbf{r}(t) = \begin{pmatrix} \cosh t \cos t \\ \cosh t \sin t \\ \sinh t \end{pmatrix}$. (Recall that $\cosh u := \dfrac{e^u + e^{-u}}{2}$ and $\sinh u := \dfrac{e^u - e^{-u}}{2}$. You may assume standard properties of these hyperbolic functions.)

(i) Compute the derivative matrix $D(F)$, i.e the gradient $\operatorname{grad} F$. (This should be a row.)
Use it to find an equation for the tangent plane to the surface $S : x^2 + y^2 - z^2 = 1$ at the point $P = (\sqrt{2}, 0, 1)$. ***(4 marks)***

(ii) Compute the derivative matrix $D(\mathbf{r})(t)$. (This velocity should be a column vector.) ***(2 marks)***

(iii) Calculate directly the product $D(F)\,(\mathbf{r}(t))\; D(\mathbf{r})(t)$, and check that it is the same as $D(F \circ \mathbf{r})\,(t)$. ***(5 marks)***

(iv) What is the relationship between $\operatorname{im}(\mathbf{r})$ and the zero set of F? ***(2 marks)***

5 Find and classify the stationary points of the function $f(x,y) = x^4 + y^4 - 36xy$. ***(12 marks)***

6 Consider the vector field $\mathbf{F} = \frac{1}{r}(x,y,z)$, where $r = \sqrt{x^2+y^2+z^2}$. Find curl $\mathbf{F}$ and div $\mathbf{F}$. ***(10 marks)***

7 Find the function $g(y)$ for which the vector field $\mathbf{v} = \begin{pmatrix} g(y) \\ xe^{2y} \end{pmatrix}$ is conservative and find the corresponding potential function for $\mathbf{v}$. Hence, evaluate the line integral

$$I = \oint_\Gamma \mathbf{v} \cdot \mathrm{d}\mathbf{r} \ ,$$

where Γ is the triangular path with vertices (0, 0), (1, 0) and (1, 1). ***(7 marks)***

8 You are given the matrix $A = \begin{pmatrix} 7 & 0 & 4 \\ 0 & -7 & -4 \\ 4 & -4 & 0 \end{pmatrix}$. You are further given that the eigenvalues of A are $0, 9$ and -9. The normalised eigenvector corresponding to the eigenvalue 0 is $\mathbf{v}_0 = \frac{1}{9}\begin{pmatrix} 4 \\ 4 \\ -7 \end{pmatrix}$. The normalised eigenvector corresponding to the eigenvalue 9 is $\mathbf{v}_9 = \frac{1}{9}\begin{pmatrix} -8 \\ 1 \\ -4 \end{pmatrix}$. Find the normalised eigenvector corresponding to the eigenvalue -9. Hence, find the orthogonal matrix P and a diagonal matrix D such that $A = PDP^\top$. ***(8 marks)***

9 (i) State Stokes' Theorem. ***(3 marks)***

(ii) Show that the integral $\oint_\mathcal{C} yzdx + xzdy + xydz$ vanishes along any closed contour $\mathcal{C}$. ***(4 marks)***

10 Let $f(x)$ be a function given by

$$f(x) = \begin{cases} 1, & -\pi < x < 0, \\ 0, & 0 < x < \pi. \end{cases}$$

Find the Fourier series for $f(x)$ in the interval $-\pi < x < \pi$.

[**Hint:** You are given that the Fourier series of a function $f(x)$, which satisfies the Dirichlet conditions on $-L \le x \le L$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left[a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right],$$

where

$$\begin{aligned} a_0 &= \frac{1}{L}\int_{-L}^{L} f(x)\mathrm{d}x \\ a_n &= \frac{1}{L}\int_{-L}^{L} f(x)\cos\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \quad n = 1, 2, 3, \ldots \\ b_n &= \frac{1}{L}\int_{-L}^{L} f(x)\sin\left(\frac{n\pi x}{L}\right)\mathrm{d}x, \quad n = 1, 2, 3, \ldots] \end{aligned}$$

(6 marks)

End of Question Paper