Data provided: formula sheet

MAS241

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2019–20**

Mathematics II (Electrical) **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

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to be completed by student

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1 (i) Consider a set of differential equations for $x(t), y(t)$

$$\begin{cases} x'(t) + 2a\,x(t) + b^2 y(t) = H(t), & (1) \\ y'(t) = x(t), & (2) \end{cases}$$

where a, b are constants and $H(t)$ denotes the Heaviside step function. The initial conditions are $x(0) = y(0) = 0$.

(a) Applying Laplace transforms to (1) and (2), write down a set of equations for $X(s), Y(s)$. Consider Laplace transforms of $x(t), y(t)$, respectively, $X(s) = \mathcal{L}(x(t)), Y(s) = \mathcal{L}(y(t))$. **(4 marks)**

(b) By eliminating $Y(s)$, obtain an equation for $X(s)$ and solve it for $X(s)$. **(2 marks)**

(c) Find $x(t)$ by inverse Laplace transform, both for $a \leq b$ and $a > b$. **(4 marks)**

(ii) For $f(t) = e^{-at^2}, g(t) = e^{-bt^2}$, where $a, b > 0$, show that

$$\mathcal{F}\{f * g(t)\} = \frac{\pi}{\sqrt{ab}} \exp\left(-\frac{a+b}{4ab}\omega^2\right)$$

in two different ways. (They can be answered independently.)

(a) Calculate the convolution $f*g(t)$ directly and apply a Fourier transform. **(5 marks)**

(b) Using the convolution theorem, calculate $\mathcal{F}\{f * g(t)\}$. **(5 marks)**

Hints: You are given that

$$\mathcal{F}\{e^{-at^2}\} = \sqrt{\frac{\pi}{a}} e^{-\frac{\omega^2}{4a}}, \quad \int_{-\infty}^{\infty} e^{-at^2} dt = \sqrt{\frac{\pi}{a}} \quad (a > 0).$$

2 Consider the function

$$f(t) = \begin{cases} 1, & (0 \leq t \leq \pi) \\ -1, & (-\pi \leq t < 0) \end{cases}$$

on the interval $[-\pi, \pi]$.

(i) Show that the Fourier series $S[f](t)$ on $[-\pi, \pi]$ is

$$S[f](t) = \frac{4}{\pi} \sum_{m=1}^{\infty} \frac{\sin(2m-1)t}{2m-1}. \qquad (1)$$

(12 marks)

(ii) Find the sum of the infinite series

$$1 - \frac{1}{3} + \frac{1}{5} - \frac{1}{7} + \ldots$$

by choosing t suitably in the above Fourier series. **(4 marks)**

(iii) For

$$g(t) = |t| \quad (-\pi \leq t \leq \pi),$$

find $S[g](t)$ by formally integrating (1) term-by-term with respect to t, or otherwise. **(4 marks)**

3 (i) (a) Let $f(x,y) = x^3 - y^3 - 3x + 12y$. Calculate the partial derivatives

$$f_x,\ f_y,\ f_{xx},\ f_{yx},\ f_{xy}.$$

(2 marks)

(b) Hence, find and classify *all* the critical points of the function $f(x,y)$.

(8 marks)

(ii) Consider the transformation of Cartesian to the polar coordinates

$$\begin{cases} x = r\cos\theta, \\ y = r\sin\theta. \end{cases}$$

(a) By chain-rule differentiation, show that for any function $f(x,y)$

$$\begin{cases} \dfrac{\partial f}{\partial r} = \quad \cos\theta\,\dfrac{\partial f}{\partial x} + \sin\theta\,\dfrac{\partial f}{\partial y}, \\ \dfrac{\partial f}{\partial \theta} = -r\sin\theta\,\dfrac{\partial f}{\partial x} + r\cos\theta\,\dfrac{\partial f}{\partial y}. \end{cases}$$

(6 marks)

(b) By (a), show that

$$\left(\frac{\partial f}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial f}{\partial \theta}\right)^2 = \left(\frac{\partial f}{\partial x}\right)^2 + \left(\frac{\partial f}{\partial y}\right)^2.$$

(4 marks)

4 (i) Let $R = \{(x,y) \mid 0 \le x \le 1,\ 0 \le y \le 1\}$. Calculate

$$\iint_R x e^{xy}\, dA.$$

(5 marks)

(ii) Let $R = \{(x,y) \mid 0 \le x \le 1,\ x^2 \le y \le x\}$. Sketch the region of integration and calculate

$$\iint_R 6x^{-2}y\, dA.$$

(7 marks)

(iii) Let $R = \{(x,y,z) \mid 0 \le x \le 1,\ 0 \le y \le 2,\ 0 \le z \le 2x + 3y\}$. Find the volume of R. **(8 marks)**

5 (i) Let $\mathbf{F} : \mathbb{R}^3 \to \mathbb{R}^3$ be the vector field defined by

$$\mathbf{F} \;=\; \left(e^{x+y},\; x^2 + y,\; xz^2 + y^2\right),$$

where $x, y, z \in \mathbb{R}$.

(a) Calculate **curl F**. **(5 marks)**

(b) Calculate **div F**. **(3 marks)**

(c) Calculate the Laplacian of **F**. **(4 marks)**

(ii) Show that **div curl** $= 0$, that is

$$\nabla \cdot (\nabla \times \mathbf{G}) = 0$$

for all vector fields $\mathbf{G} : \mathbb{R}^3 \to \mathbb{R}^3$. **(4 marks)**

(iii) Let $f(x, y) = x^2y + xy + 1$. Calculate the directional derivative of $f(x, y)$ at $(x, y) = (1, 0)$ in the direction $v = (2, 1)$. **(4 marks)**

End of Question Paper

MAS241 FORMULA SHEET

Laplace transform:

The Laplace transform of a function $f(t)$ is given by:

$$\mathcal{L}\{f(t)\}(s) := \int_0^\infty e^{-st} f(t) dt.$$

Properties of the Laplace transform: $\mathcal{L}\{f(t)\} = F(s)$ in the following table.

$\mathcal{L}\{af(t) + bg(t)\} = a\mathcal{L}\{f(t)\} + b\mathcal{L}\{g(t)\}$	linearity
$\mathcal{L}\{f'(t)\} = sF(s) - f(0)$	differentiation w.r.t. t
$\mathcal{L}\{f''(t)\} = s^2F(s) - sf(0) - f'(0)$	second differentiation w.r.t. t
$\mathcal{L}\{e^{-kt}f(t)\} = F(k+s)$	frequency shift
$\mathcal{L}\{f(t-a)H(t-a)\} = e^{-as}F(s)$ (for $a > 0$)	time shift
$\mathcal{L}\{f(at)\} = \frac{1}{a}F(\frac{s}{a})$ (for $a > 0$)	scaling
$\mathcal{L}\{f * g(t)\} = \mathcal{L}\{f(t)\}\mathcal{L}\{g(t)\}$ (for $f(t), g(t)$ causal)	convolution

Table of standard Laplace transforms:

$f(t)$	$\mathcal{L}\{f(t)\}(s)$	Region of validity
t^n (for $n \geq 0$)	$\frac{n!}{s^{n+1}}$	$Re(s) > 0$
$\sin(kt)$	$\frac{k}{s^2+k^2}$	$Re(s) > 0$
$\cos(kt)$	$\frac{s}{s^2+k^2}$	$Re(s) > 0$
$H(t-T)$ (for $T \geq 0$)	$\frac{e^{-sT}}{s}$	$Re(s) > 0$
$\delta(t-T)$ (for $T \geq 0$)	e^{-sT}	$s \in \mathbb{C}$

Fourier transform:

The Fourier transform and inverse Fourier transforms are given by:

$$\mathcal{F}\{f(t)\}(\omega) = F(\omega) := \int_{-\infty}^{\infty} f(t)e^{-j\omega t} dt, \qquad f(t) = \mathcal{F}^{-1}\{F(\omega)\} = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega)e^{j\omega t} d\omega.$$

Properties of the Fourier transform: $\mathcal{F}\{f(t)\} = F(\omega)$ in the following table:

$\mathcal{F}\{e^{j\theta t} f(t)\} = F(\omega - \theta)$	frequency shift
$\mathcal{F}\{f(t - T)\} = e^{-j\omega T} F(\omega)$	time shift
$\mathcal{F}\{f^{(n)}(t)\} = (j\omega)^n F(\omega)$	differentiation
$\mathcal{F}\{F(t)\} = 2\pi f(-\omega)$	symmetry
$\mathcal{F}\{f(at)\} = \frac{1}{\lvert a \rvert} F(\frac{\omega}{a})$	scaling
$\mathcal{F}\{f * g(t)\} = \mathcal{F}\{f(t)\}\mathcal{F}\{g(t)\}$	convolution

Table of standard Fourier transforms:

$f(t)$	$\mathcal{F}\{f(t)\}(\omega)$
$e^{-a\lvert t \rvert}$ (for $a > 0$)	$\frac{2a}{a^2+\omega^2}$
$\mathrm{rect}_T(t)$	$\mathrm{sinc}(\frac{T\omega}{2})$
1	$2\pi\delta(\omega)$

Fourier series:

The Fourier series of a periodic function $f(t)$ with fundamental period T is given by

$$S[f] = \frac{a_0}{2} + \sum_{n=1}^{\infty} \Big(a_n \cos(\omega_n t) + b_n \sin(\omega_n t) \Big)$$

where

$$\omega_n = \frac{2\pi n}{T}, \qquad a_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \cos(\omega_n t) dt, \qquad b_n = \frac{2}{T} \int_{-T/2}^{T/2} f(t) \sin(\omega_n t) dt.$$

Coordinate systems:

Cylindrical polar coordinates

$$(x, y, z) = (r\cos(\theta), r\sin(\theta), z)$$
$$(r, \theta, z) = \Big(\sqrt{x^2 + y^2}, \arctan(\tfrac{y}{x}), z \Big)$$
$$dV = r dr d\theta dz.$$

Spherical polar coordinates

$$(x, y, z) = (\rho\sin(\phi)\cos(\theta), \rho\sin(\phi)\sin(\theta), \rho\cos(\phi))$$
$$(\rho, \theta, \phi) = \Big(\sqrt{x^2 + y^2 + z^2}, \arctan(\tfrac{y}{x}), \arccos(\tfrac{z}{\rho}) \Big)$$
$$dV = \rho^2 \sin(\phi) d\rho d\phi d\theta.$$