Data provided: Formula sheet

MAS248

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2019–20**

MATHEMATICS III (CHEMICAL) **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.
The paper is marked out of a total of 60 marks.

1 Show that the partial differential equation

$$\frac{\partial^2 y}{\partial t^2} - c^2 \frac{\partial^2 y}{\partial x^2} = 0$$

has solutions of the form $y(x,t) = f(x + \lambda t)$ for arbitrary twice differentiable functions f, provided that $\lambda = -c$ or $\lambda = c$, where c is a positive constant.

(4 marks)

Write down the general solution of the partial differential equation. ***(1 mark)***

Use D'Alembert's method to find the particular solution for $y(x,t)$ for $t \geq 0$ that satisfies the conditions

$$y(x,0) = u_0 \cos\left(\frac{x}{L}\right)$$

and

$$\frac{\partial y}{\partial t}(x,0) = \frac{u_0 c}{L} \sin\left(\frac{x}{L}\right),$$

where $u_0 > 0$ and $L > 0$ are positive constants. ***(10 marks)***

2 (i) Write down the iteration formula for the Newton-Raphson method. Starting with an initial guess of $x_0 = 1.1$, use the Newton-Raphson method to find the seventh root of 3 correct to 4 decimal places.

(4 marks)

(ii) A surveyor estimates the area, A, of a triangular plot of land using the formula

$$A = \frac{1}{2}ab\sin C,$$

where a and b are the lengths of the two sides and C is the included angle. If the sides are measured to an accuracy of 2%, use the formula for small increments to calculate approximately the maximum percentage error in A. You may assume that C is measured accurately. ***(7 marks)***

(iii) Verify that the equation

$$\frac{\partial^2 z}{\partial x^2} + \frac{\partial^2 z}{\partial y^2} = 0$$

is satisfied by the function $z(x, y) = \ln(x^2 + y^2)$. ***(4 marks)***

3 (i) (a) Determine whether or not the vector $\mathbf{A} = (4xy - z^3, 2x^2, -3xz^2)$ is irrotational. ***(1 mark)***

(b) If r is the distance of a point (x, y, z) from the origin O, show that

$$\nabla\left(\frac{1}{r}\right) = -\frac{\mathbf{r}}{r^3}.$$

(3 marks)

Hence show that

$$\nabla \cdot \left(\nabla\left(\frac{1}{r}\right)\right) = 0.$$

(4 marks)

(ii) (a) The probability density function of a random variable, X, is given by

$$f(x) = \begin{cases} \dfrac{ae^{-ax}}{1 - e^{-aH}} & \text{for} \quad 0 \le x \le H, \\ \\ 0 & \text{otherwise,} \end{cases}$$

where a and H are constants.
Calculate the mean value of the random variable, X, giving your answer in the form

$$b - \frac{ce^{-aH}}{1 - e^{-aH}},$$

where b and c are constants that should be determined in terms of a and H respectively. ***(5 marks)***

(b) A continuous random variable Y satisfies the normal distribution and has probability density function

$$f(y) = \frac{1}{9\sqrt{2\pi}} \exp\left(-\frac{(y-7)^2}{162}\right), \quad \text{for} \quad -\infty < y < \infty.$$

What are its mean and variance? ***(2 marks)***

4 (i) Determine whether each of the following functions is even, odd, or neither even nor odd.

(a) $2x^3 + x^2$

(b) $3x^6 - 5x^4 + 6x^2 - 1$

(c) $\sin x \cosh x$ ***(3 marks)***

(ii) A periodic function, $f(t)$, with period 2 is defined by

$$f(t) = \begin{cases} t - 0.5 & \text{for} \quad 0 \leq t < 1, \\ \\ 1.5 - t & \text{for} \quad 1 \leq t < 2. \end{cases}$$

(a) Sketch a graph of the function $f(t)$ for values of t from $t = -4$ to $t = 4$. ***(2 marks)***

(b) Find the Fourier series for $f(t)$. ***(6 marks)***

(iii) Without performing integration, derive the Fourier coefficients for

(a)

$$g(x) = \sin x \cos x.$$

(1 mark)

(b)

$$p(x) = \tan x \sin 2x.$$

(3 marks)

End of Question Paper

Formula Sheet

Fourier Series

Suppose that $f(x)$ is defined on the interval $-L \leq x \leq L$. The Fourier series for $f(x)$ is given by

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos\frac{n\pi x}{L} + b_n \sin\frac{n\pi x}{L} \right)$$

where

$$a_n = \frac{1}{L}\int_{-L}^{L} f(x)\cos\frac{n\pi x}{L}dx, \qquad n = 0, 1, 2, ...,$$

$$b_n = \frac{1}{L}\int_{-L}^{L} f(x)\sin\frac{n\pi x}{L}dx, \qquad n = 1, 2, 3, ...$$

On the interval $0 \leq x \leq L$ the Fourier cosine series for $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{L}, \qquad a_n = \frac{2}{L}\int_{0}^{L} f(x)\cos\frac{n\pi x}{L}dx$$

and the Fourier sine series is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{L}, \qquad b_n = \frac{2}{L}\int_{0}^{L} f(x)\sin\frac{n\pi x}{L}dx$$

Gradient of a Scalar Field

The gradient of the scalar field $\phi(x, y, z)$ is given by

$$\nabla\phi = \text{grad } \phi = \left(\frac{\partial\phi}{\partial x}, \frac{\partial\phi}{\partial y}, \frac{\partial\phi}{\partial z} \right).$$

Chain Rule

1 If $z = f(x, y)$, where $x = x(t)$, $y = y(t)$, then

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}.$$

2 If $z = f(x, y)$, where $x = x(u, v)$, $y = y(u, v)$, then

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}, \quad \frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}.$$

3 If $z = f(u, v)$, where $u = u(x, y)$, $v = v(x, y)$, then

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial x} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial x}, \quad \frac{\partial z}{\partial y} = \frac{\partial z}{\partial u}\frac{\partial u}{\partial y} + \frac{\partial z}{\partial v}\frac{\partial v}{\partial y}.$$

Maxima and Minima

1 The function $f(x, y)$ has a stationary point at (x_0, y_0) if

$$f_x = f_y = 0 \quad \text{at} \quad (x_0, y_0).$$

2 At (x_0, y_0), the function $f(x, y)$ has:

(i) a minimum if

$$f_{xx}f_{yy} - f_{xy}^2 > 0 \quad \text{and} \quad f_{xx} > 0 \quad \text{at} \quad (x_0, y_0),$$

(ii) a maximum if

$$f_{xx}f_{yy} - f_{xy}^2 > 0 \quad \text{and} \quad f_{xx} < 0 \quad \text{at} \quad (x_0, y_0),$$

(iii) a saddle point if

$$f_{xx}f_{yy} - f_{xy}^2 < 0 \quad \text{at} \quad (x_0, y_0).$$