Data provided: Formula sheet

MAS252

The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS **Autumn Semester 2019–20**

Further Civil Engineering Mathematics and Computing **2 hours**

Attempt all the questions. The allocation of marks is shown in brackets.

1 (i) Find and classify all the stationary points of the function

$$f(x, y) = \frac{1}{3}x^3 + \frac{1}{3}y^3 - x^2 - y^2.$$

(14 marks)

(ii) Find all the first order partial derivatives of the function

$$z(x, y) = \sqrt{x^2 + \ln(5x - 3y^2)},$$

and compute these derivatives at $x = 1$, $y = 1$. ***(6 marks)***

(iii) The Fourier series of a function is given by

$$\sum_{n=1}^{\infty} \frac{n + a}{n^3 + an + 3} \sin nx,$$

where a is a constant parameter. Find the value of a in order that the leading harmonics $n = 1$ and $n = 2$ have the amplitudes in the ratio $2 : 1$. What is the amplitude of the next harmonic? ***(5 marks)***

2 A rectangular elastic membrane has a steady state stress distribution described by the Laplace equation

$$\frac{\partial^2 w}{\partial x^2} + \frac{\partial^2 w}{\partial y^2} = 0,$$

where $w(x, y)$ is the displacement from its equilibrium position and the membrane is stretched according to the boundary conditions

$$w(0, y) = 0, \qquad 0 \le y \le b,$$

$$w(x, 0) = 0, \qquad 0 \le x \le a,$$

$$w(x, b) = 0, \qquad 0 \le x \le a,$$

$$w(a, y) = w_0 \sin\frac{\pi y}{b}, \qquad 0 \le y \le b.$$

(i) Sketch the domain of interest, clearly marking the values of the boundary conditions ***(3 marks)***

(ii) Show that the solution of the Laplace equation by means of the method of separation of variables is

$$w(x, y) = \frac{w_0}{\sinh \pi a/b} \sinh\frac{\pi x}{b} \sin\frac{\pi y}{b}.$$

(22 marks)

3 (i) Show that the Fourier series representation of the function $f(t) = t^2$ over the interval $-\pi \le t \le \pi$ is

$$t^2 = \frac{\pi^2}{3} + 4\sum_{n=1}^{\infty} \frac{(-1)^n}{n^2} \cos nt.$$

By integrating all the terms in this expression from $t = 0$ to $t = x$, obtain a Fourier series for $x^3 - \pi^2 x$. ***(15 marks)***

(ii) Find the the first four non-zero terms of the series solution of the differential equation

$$xy'' - (x + 1)y' + y = 0; \quad y = y(x),$$

subject to the conditions $y(1) = 1$ and $y'(1) = -1$. Hence, use the series representation to calculate $y(0.9)$. Give your answer correct to *four* decimal places. ***(10 marks)***

4 (i) Values of $y(x)$ at $x = 2$ determined using the fourth-order Runge-Kutta method in conjunction with an ordinary differential equation with two different step-lengths, h, are given in the following table

h	$y(2)$
0.2	3.40978
0.4	3.39278

Use this data to estimate a value for h which will ensure that the error in the calculated value of $y(2)$ using a fourth-order Runge-Kutta method does not exceed 10^{-4}. Give your answer correct to 4 decimal places.

(7 marks)

(ii) Let us consider the function

$$f(x, y) = (2x + y)e^{xy}.$$

Find all values of x such that the equality

$$\left.\frac{\partial f(x, y)}{\partial x}\right|_{y=1} = \left.\frac{\partial f(x, y)}{\partial y}\right|_{y=1}$$

holds. ***(6 marks)***

(iii) The temperature, T, measured at the point $P(x, y, z)$ of a steel beam is given in a rectangular coordinate system by

$$T = \left[2x^2 + \ln(xy) + 1/z\right]^{1/2}.$$

Use the small error formula to estimate the change in temperature if the measurement is moved from the position $(6, 3, 2)$ to $(6.1, 3.3, 1.98)$. In your calculation work throughout to *four* decimal places. ***(12 marks)***

End of Question Paper

Formula sheet

- **Trigonometric identities**

$$\cos(A \pm B) = \cos(A)\cos(B) \mp \sin(A)\sin(B)$$

$$\sin(A \pm B) = \sin A \cos B \pm \cos A \sin B$$

- **The local truncation error in the case of the 4th order Runge-Kutta method is given by**

$$Y(x) - y(x) = Ch^4$$

where $Y(x)$ is the exact value, $y(x)$ is the estimated numerical value, C is a constant and h is the step size used in the numerical scheme.

- **Chain rule**

If $z = f(x, y)$, where x and y are both functions of t, so that $x = x(t)$ and $y = y(t)$ we have

$$\frac{dz}{dt} = \frac{\partial z}{\partial x}\frac{dx}{dt} + \frac{\partial z}{\partial y}\frac{dy}{dt}$$

If $z = f(x, y)$ and both x and y are functions of u and v, so that $x = x(u, v)$ and $y = y(u, v)$ then we have

$$\frac{\partial z}{\partial u} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial u}$$

$$\frac{\partial z}{\partial v} = \frac{\partial z}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial z}{\partial y}\frac{\partial y}{\partial v}$$

- **Fourier series**

If the function $f(x)$ is defined over the interval $-l \leq x \leq l$, then the Fourier series of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos\frac{n\pi x}{l} + b_n \sin\frac{n\pi x}{l}\right)$$

where

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x)\cos\frac{n\pi x}{l}\ dx, \quad (n = 0, 1, 2, \dots)$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x)\sin\frac{n\pi x}{l}\ dx \quad (n = 1, 2, 3, \dots)$$

If the function $f(x)$ is defined over the interval $0 \leq x \leq l$, then the Fourier cosine series of $f(x)$ is

$$f(x) = \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos\frac{n\pi x}{l}, \qquad a_n = \frac{2}{l}\int_0^l f(x)\cos\frac{n\pi x}{l}\;dx, \quad (n = 0, 1, 2, \dots)$$

while the sine series of $f(x)$ is

$$f(x) = \sum_{n=1}^{\infty} b_n \sin\frac{n\pi x}{l}, \qquad b_n = \frac{2}{l}\int_0^l f(x)\sin\frac{n\pi x}{l}\;dx \quad (n = 1, 2, 3, \dots)$$

- **the orthogonality of the sine function can be defined as**

$$\int_0^L \sin\left(\frac{n\pi x}{L}\right)\sin\left(\frac{m\pi x}{L}\right) dx = \begin{cases} 0 & \textbf{if } m \neq n \\ L/2 & \textbf{if } m = n \end{cases}$$