The University Of Sheffield.

SCHOOL OF MATHEMATICS AND STATISTICS — Autumn Semester 2019-2020

Mathematics for Engineering Modelling — 2 hours

Answer ***four*** *questions. You are advised* ***not*** *to answer more than four questions: if you do, only your best four will be counted.*

Please leave this exam paper on your desk
Do not remove it from the hall

Registration number from U-Card (9 digits)
to be completed by student

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1 (i) Derive the Maclaurin series for $\sin(x)$ up to the x^5 term and for $\cos(x)$ up to the x^4 term. *Use the series* to show that, up to and including terms in x^5,

$$\sin^2(x) + \cos^2(x) = 1.$$

(12 marks)

(ii) Evaluate

$$\lim_{x\to 0} \frac{\sin^2(x)}{1-\cos(x)}.$$

(3 marks)

(iii) *Derive* the partial sum s_n to n terms of the infinite series

$$a + ar + ar^2 + ar^3 + \cdots + ar^{n-1} + \cdots \qquad (r \neq 1).$$

Hence show that the series is convergent for $| r |< 1$ and find the sum.

Find the sum of the infinite geometric series

$$1 + 2x + 4x^2 + 8x^3 + 16x^4 + \cdots,$$

and state the value of the radius of convergence, R, of the series. Hence find, for $| x |< R$, the sum of the infinite series

$$2 + 8x + 24x^2 + 64x^3 + \cdots.$$

(10 marks)

2 (i) Show that the Fourier *sine* series of $\cos(x)$, $0 < x < \pi$, is

$$\frac{8}{\pi}\sum_{m=1}^{\infty} \frac{m}{4m^2-1}\sin(2mx).$$

(16 marks)

(ii) For $-2\pi < x < 2\pi$ sketch the graph of the sum of the series.

(4 marks)

(iii) Deduce that

$$\frac{1}{4\cdot 1^2 - 1} - \frac{3}{4\cdot 3^2 - 1} + \frac{5}{4\cdot 5^2 - 1} - \cdots = \frac{\pi\sqrt{2}}{16}.$$

(5 marks)

3 (i) Find *by integration* the Laplace transform $F(s)$ of t. (You may assume that $s > 0$.) State the Shift Theorem and use it to deduce the Laplace transform of te^{2t}.

(3 marks)

(ii) *Verify* that

$$\frac{1}{s^2(s^2-3s+2)} = \frac{1}{4}\left\{\frac{1}{s-2} - \frac{4}{s-1} + \frac{3}{s} + \frac{2}{s^2}\right\}.$$

(4 marks)

(iii) Use the method of Laplace transforms to solve the ordinary differential equation

$$\frac{d^2y}{dt^2} - 3\frac{dy}{dt} + 2y = t,$$

subject to the initial conditions $y(0) = 0$, $dy/dt = 0$ at $t = 0$.

(9 marks)

(iv) A particle is attached to one end of a light spring, whose other end is fixed. The particle moves in a vertical line and its equation of motion is

$$\frac{d^2x}{dt^2} + \omega^2 x = V\delta(t-T),$$

where x is the distance below its equilibrium position, $\delta(t-T)$ is the Dirac delta function and ω, V are positive constants. Find $x(t)$ using Laplace transforms, given that $x(0) = 0$, $dx/dt = V/2$ at $t = 0$.

Sketch $x\omega/V$ against ωt for $0 \le \omega t \le 3\pi$ for the case $\omega T = \pi$. Give a very brief physical description of the motion and your solution.

(9 marks)

4 (i) *Derive* d'Alembert's general solution for the one-dimensional wave equation

$$\frac{\partial^2 \phi}{\partial t^2} = c^2 \frac{\partial^2 \phi}{\partial x^2}$$

on $-\infty < x < \infty$ for $t \geq 0$, assuming c is a constant.

(13 marks)

(ii) Find $\phi(x, t)$, given that $c = 1$ and at $t = 0$

$$\phi(x, 0) = \sin kx, \qquad \frac{\partial \phi}{\partial t} = -k \cos kx,$$

where k is a constant.

(7 marks)

(iii) Give a physical interpretation of your solution. Further, explain why the solution, subject to the initial conditions in (ii), cannot (or can) be a standing wave.

(5 marks)

5 (i) (i) Let R be the rectangular region bounded by the lines $x = -1$, $x = 2$, $y = 0$, and $y = 2$. Find

$$\int\int_R x^2 y dx dy.$$

(5 marks)

(ii) Let R be the region bounded by $y = x^2$ and $y = x + 6$. Sketch the region R and evaluate

$$\int\int_R x dx dy.$$

(8 marks)

(iii) By reversing the order of integration, evaluate

$$\int_0^9 \int_{\sqrt{y}}^3 \sin(\pi x^3) dx dy.$$

(12 marks)

End of Question Paper

For use with MAS253 first semester examination

Formulae for use in L2 Mechanical Engineering Mathematics Examination

These results may be quoted without proof unless proofs are asked for in the question.

Trigonometry

$$\sin 2P = 2\sin P \cos P,$$

$$\cos 2P = \cos^2 P - \sin^2 P = 2\cos^2 P - 1 = 1 - 2\sin^2 P,$$

$$\cos P \cos Q = \frac{1}{2}\{\cos(P+Q) + \cos(P-Q)\},$$

$$\sin P \sin Q = -\frac{1}{2}\{\cos(P+Q) - \cos(P-Q)\},$$

$$\sin P \cos Q = \frac{1}{2}\{\sin(P+Q) + \sin(P-Q)\}.$$

Geometric progression

The partial sum to n terms of

$$a + ar + ar^2 + \ldots\ldots + ar^{n-1} + \ldots\ldots$$

is

$$S_n = a(1-r^n)/(1-r), \quad r \neq 1 .$$

Taylor Series for functions of one variable (x)

The Taylor series of $f(x)$ about $x = a$ is

$$f(x) = f(a) + f'(a)(x-a) + \frac{1}{2!}f''(a)(x-a)^2 + \ldots\ldots$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(a)}{n!}(x-a)^n$$

The Maclaurin series of $f(x)$ is the special case of the Taylor series when $a = 0$:

$$f(x) = f(0) + f'(0)\,x + \frac{1}{2!}f''(0)\,x^2 + \ldots\ldots$$

$$= \sum_{n=0}^{\infty} \frac{f^{(n)}(0)}{n!}x^n$$

Important examples of Maclaurin series are:

$$e^x = 1 + x + \frac{1}{2!}x^2 + \ldots \qquad (R \text{ is infinite})$$

$$\sin x = x - \frac{1}{3!}x^3 + \frac{1}{5!}x^5 - \ldots. \qquad (R \text{ is infinite})$$

$$\cos x = 1 - \frac{1}{2!}x^2 + \frac{1}{4!}x^4 - \ldots. \qquad (R \text{ is infinite})$$

$$\ln(1+x) = x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \ldots \qquad (R = 1)$$

$$(1+x)^\alpha = 1 + \alpha x + \frac{\alpha(\alpha-1)}{2!}x^2 + \ldots. \qquad (R = 1)$$

R is the radius of convergence.

Binomial Theorem

$$(1+x)^n = 1 + nx + \frac{n(n-1)}{1.2}x^2 + \frac{n(n-1)(n-2)}{1.2.3}x^3 + \ldots..$$

If n is positive and integer, series terminates.

If n is negative or non-integer (or both), the series is an infinite series with radius of convergence, $R = 1$.

Fourier Series

The Fourier series of $f(x)$ for $-l < x < l$ is

$$\frac{1}{2}a_0 + \sum_{n=1}^{\infty}\left(a_n \cos\left(\frac{n\pi x}{l}\right) + b_n \sin\left(\frac{n\pi x}{l}\right)\right)$$

where

$$a_0 = \frac{1}{l}\int_{-l}^{l} f(x)\, dx ,$$

$$a_n = \frac{1}{l}\int_{-l}^{l} f(x) \cos\left(\frac{n\pi x}{l}\right) dx , \quad n = 1, 2, \ldots$$

$$b_n = \frac{1}{l}\int_{-l}^{l} f(x) \sin\left(\frac{n\pi x}{l}\right) dx , \quad n = 1, 2, \ldots$$

Laplace Transform

The Laplace Transform of $f(t)$ is

$$F(s) = L(f(t)) = \int_0^\infty e^{-st} f(t)dt .$$

For special cases, see later page.

Partial Differentiation

$$\delta F = F(x+\delta, y+\varepsilon) - F(x,y) \cong \delta \frac{\partial F}{\partial x} + \varepsilon \frac{\partial F}{\partial y}$$

Chain Rules:

1. Suppose that $F = F(x,y)$ and that x and y are functions of t,
   i.e. $x = x(t), y = y(t)$, then
$$\frac{dF}{dt} = \frac{\partial F}{\partial x}\frac{dx}{dt} + \frac{\partial F}{\partial y}\frac{dy}{dt}.$$
2. Suppose that $F = F(x,y)$ and that x and y are functions of the variables u and v, i.e. $x = x(u,v), y = y(u,v)$, then
$$\frac{\partial F}{\partial u} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial u} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial u}; \qquad \frac{\partial F}{\partial v} = \frac{\partial F}{\partial x}\frac{\partial x}{\partial v} + \frac{\partial F}{\partial y}\frac{\partial y}{\partial v}.$$

Taylor Series for functions of two variables (x, y)

The Taylor series of $f(xy)$ about $x = a, y = b$ is

$$f(xy) = f(a,b) + \{(x-a) f_x(a,b) + (y-b) f_y(a,b)\} +$$
$$+ \frac{1}{2!}\Big\{(x-a)^2 f_{xx}(a,b) + 2(x-a)(y-b) f_{xy}(a,b) +$$
$$+ (y-b)^2 f_{yy}(a,b)\Big\} +$$
$$+ \ldots$$

Here $f_x = \dfrac{\partial f}{\partial x}$ etc.

Partial Differential Equations (2 independent variables)

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$ Laplace's equation

$$\frac{\partial^2 V}{\partial x^2} = \frac{1}{K}\frac{\partial V}{\partial t}$$ Heat conduction (or diffusion) eqn. equation

$$\frac{\partial^2 V}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 V}{\partial t^2}$$ Wave equation

General Solution of ODEs

$$X'' = -\omega^2 X \Rightarrow X(x) = A\cos\omega x + B\sin\omega x$$
$$X'' = \omega^2 X \Rightarrow X(x) = C\cosh\omega x + D\sinh\omega x$$
$$\text{or } Ee^{\omega x} + Fe^{-\omega x}$$
$$T' = kT \Rightarrow T(t) = Ae^{kt}$$

Table of Laplace Transforms	
$f(t)$	$F(s) = L(f(t))$
$f(t)$	$F(s)$
$f'(t)$	$sF(s) - f(0)$
$f''(t)$	$s^2 F(s) - sf(0) - f'(0)$
$f^{(iv)}(t)$	$s^4 F(s) - s^3 f(0) - s^2 f'(0) - sf''(0) - f'''(0)$
1	$1/s$
t	$1/s^2$
$t^{n-1}/(n-1)!\,(n \geq 1)$	$1/s^n$
e^{at}	$\frac{1}{s-a}$
$\frac{1}{a}\sin at$	$\frac{1}{s^2+a^2}$
$\cos at$	$\frac{s}{s^2+a^2}$
$\frac{1}{a}\sinh at$	$\frac{1}{s^2-a^2}$
$\cosh at$	$\frac{s}{s^2-a^2}$
$\frac{\sin at - at\cos at}{2a^3}$	$\frac{1}{(s^2+a^2)^2}$
$\frac{t\sin at}{2a}$	$\frac{s}{(s^2+a^2)^2}$
$e^{at} f(t)$	$F(s-a)$, where $F(s) = L(f(t))$
$\delta(t)$	1
$\delta(t-a)$	e^{-as}
$u(t-a)$	e^{-as}/s
$u(t-a)f(t-a)$	$e^{-as}F(s)$, where $F(s) = L(f(t))$